Generalized spherical distributions

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Outline

1. Generalized spherical distributions

2. Ratios of stable r.v.

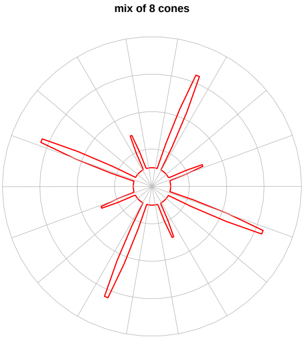
Generalized spherical distributions

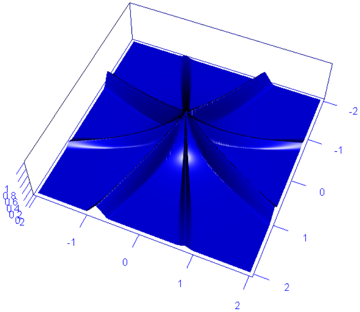
Let $\mathbf{X}$ have a density $f(x)$ on $\mathbb{R}^d$ and let $\mathbb{S}$ be the unit sphere $\{\mathbf{x} : |\mathbf{x}| = 1\}$, $\mathbb{B}$ be the unit ball $\{\mathbf{x} : |\mathbf{x}| \leq 1\}$. A distribution is spherically distributed if $f(\cdot)$ is constant on each sphere $r\mathbb{S}$, $r > 0$.

A distribution is generalized spherical if there is a curve/surface $\mathbb{S}_*$ with $f(\cdot)$ being constant on all multiples $r\mathbb{S}_*$, $r > 0$.

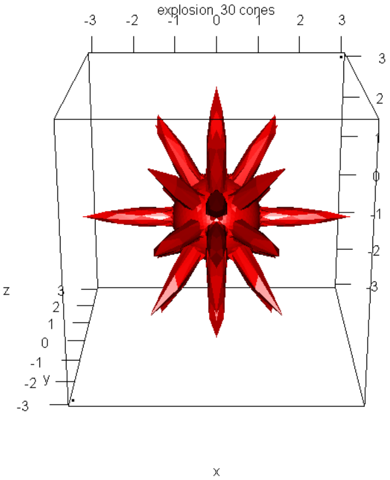
Goal: to have flexible program to work with large classes of such distributions in $d-$dimensions.

Star shaped distribution





Start shaped contour in 3D

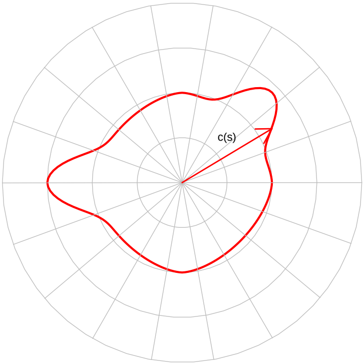


Previous work

Fernandez, Osiewalski and Steel (1995) gave idea, Arnold, Castillo and Sarabia (2008) extended some and advocated modeling data with these.

We will start with a contour/surface given by a a polar representation:

$$\mathbb{S}_* = \{c(\mathbf{s})\mathbf{s} : \mathbf{s} \in \mathbb{S}\}$$



Radial function and density

Let $g(r) \geq 0$ be an integrable function on $[0, \infty)$, it will determine the radial decay of the density. Using the contour function $c(\mathbf{s})$ and the radial function, define

$$f(\mathbf{x}) = \begin{cases} g\left(\frac{|\mathbf{x}|}{c(\mathbf{x}/|\mathbf{x}|)}\right) & |\mathbf{x}| > 0 \\ g(0) & |\mathbf{x}| = 0. \end{cases} \qquad (1)$$

With suitable integrability conditions, this gives a density on $\mathbb{R}^d$.

Fernandez, Osiewalski and Steel (1995) started with a homogeneous function $v(\mathbf{x})$ on $\mathbb{R}^d$ ($v(r\mathbf{x}) = rv(\mathbf{x})$) and defined $\mathbb{B}_* = \{\mathbf{x} \in \mathbb{R}^d : v(\mathbf{x}) \leq 1\}$. If $\mathbb{B}_*$ is convex and symmetric, then $v(\cdot)$ is a norm on $\mathbb{R}^d$ with unit ball $\mathbb{B}_*$ and unit sphere given by it's boundary $\mathbb{S}_* = \{\mathbf{x} \in \mathbb{R}^d : v(\mathbf{x}) = 1\}$. In general, $v(\cdot)$ is not a norm, but we may still call $\mathbb{S}_*$ a "unit ball". In their approach, the density

$$f(\mathbf{x}) = g(v(\mathbf{x}))$$

is called a v-spherical density. A.K.A. homothetic distributions.

In terms of the contour function, $v(\mathbf{x}) = |\mathbf{x}|/c(\mathbf{x}/|\mathbf{x}|)$. We find using the contour function as the starting point a more intuitive approach: it defines the unit ball, which defines the level curves.

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For (1) to be a proper density, it is required that

$$k_*^{-1} := \int_{\mathbb{S}} c^d(\mathbf{s})d\mathbf{s} \in (0, \infty) \tag{2}$$

and

$$\int_0^\infty r^{d-1} g(r) dr = k_*.$$

We will assume $c(\mathbf{s})$ is continuous and bounded away from 0 on compact $\mathbb{S}$, so the k_* is finite. An easy way to guarantee the second condition is to start with a univariate density $h(r)$ on $[0, \infty)$ and define $g(r) = k_* r^{1-d} h(r)$.

Specifying the contour

We wanted flexible, parametric families of generalized spherical distributions that worked in arbitrary dimensions that included most of the cases described in the earlier work. We allow contour functions of the form

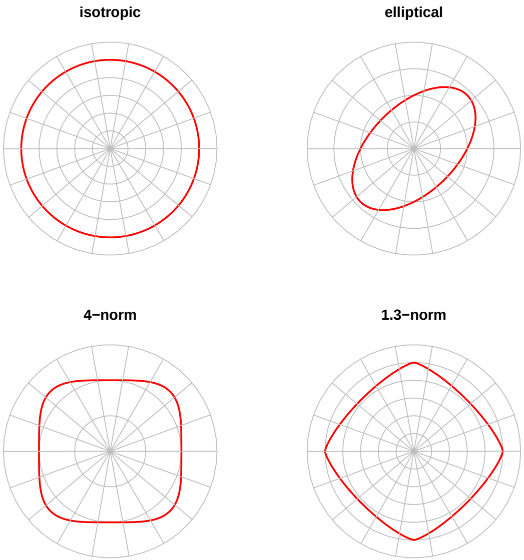
$$c(\mathbf{s}) = \sum_{j=1}^{N_1} a_j c_j(\mathbf{s}) + \frac{1}{\sum_{j=1}^{N_2} a_j^* c_j^*(\mathbf{s})},$$

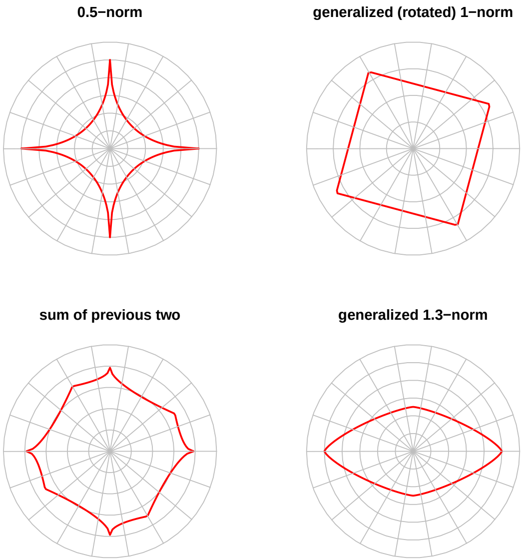
where $c_j > 0$, $c_j^* > 0$, and $c_j(\cdot)$ and/or $c^*(\cdot)$ are one of the cases discussed below.

- $c(\mathbf{s}) = 1$, which makes $\mathbb{S}_*$ the Euclidean ball.
- $c(\mathbf{s}) = c(\mathbf{s}|\boldsymbol{\mu}, \theta)$ is a cone with peak 1 at center $\boldsymbol{\mu} \in \mathbb{S}$ and height 0 at the base given by the circle $\{\mathbf{x} \in \mathbb{S} : \boldsymbol{\mu} \cdot \mathbf{x} = \cos\theta\}$. It is assumed that $|\theta| \leq \pi/2$.
- $c(\mathbf{s}) = c(\mathbf{s}|\boldsymbol{\mu}, \sigma) = \exp(-t(\mathbf{s})^2/(2\sigma^2))$ is a Gaussian bump centered at location $\boldsymbol{\mu} \in \mathbb{S}$ and "standard deviation" $\sigma > 0$. Here $t(\mathbf{s})$ is the distance between $\boldsymbol{\mu}$ and the projection of $\mathbf{s} \in \mathbb{S}$ linearly onto the plane tangent to $\mathbb{S}$ at $\boldsymbol{\mu}$.

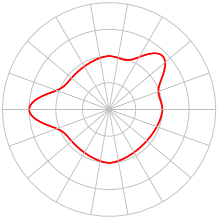
- $r^*(\mathbf{s}) = ||\mathbf{s}||_{\ell^p(\mathbb{R}^d)}$, $p > 0$.
- $r^*(\mathbf{s}) = ||A\mathbf{s}||_{\ell^p(\mathbb{R}^m)}$, $p > 0$, A an $(m \times d)$ matrix. This allows a generalized p-norm. If A is $d \times d$ and orthogonal, then the resulting contour will be a rotation of the standard unit ball in ℓ^p. If A is $d \times d$ and not orthogonal, then the contour will be sheared. If $m > d$, it will give the ℓ^p norm on $\mathbb{R}^m$ of $A\mathbf{s}$.
- $r^*(\mathbf{s}) = (\mathbf{s}^T A\mathbf{s})^{1/2}$, where A is a positive definite $(d \times d)$ matrix. Then the level curves of the distribution are ellipses.

Sums of the last three types allow us to consider contours that are familiar unit balls, or generalized unit balls, or sums of such shapes. Sums of the first three types allow us to describe star shaped contours. Combinations allow more general shapes, see the following plots.

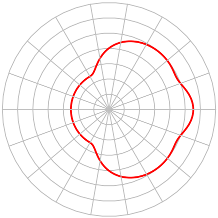

isotropic
elliptical
4−norm
1.3−norm


0.5−norm
generalized (rotated) 1−norm
sum of previous two
generalized 1.3−norm

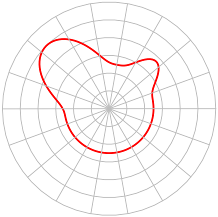
blob #1



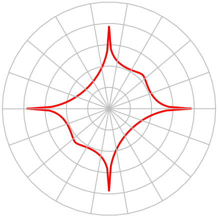
blob #2

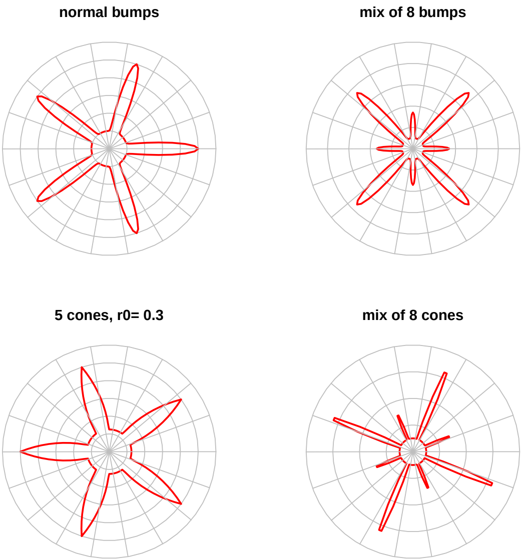


blob #3



blob #4




normal bumps
mix of 8 bumps
5 cones, r0= 0.3
mix of 8 cones

The radial function

Generally want $g(r)$ to be decreasing so that $f(\mathbf{x})$ is unimodal on $\mathbb{R}^d$. Here are two accessible classes, defined in terms of univariate r.v. $R \geq 0$, which has pdf $h(r)$.

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Generally want $g(r)$ to be decreasing so that $f(\mathbf{x})$ is unimodal on $\mathbb{R}^d$. Here are two accessible classes, defined in terms of univariate r.v. $R \geq 0$, which has pdf $h(r)$.

- $R \sim$Gamma$(d+a+1)$, then $g(r) = k_* r^{1-d} h(r)$ is a constant times a Gamma(a). If $a \in (0,1]$, then $g(r)$ is decreasing. Is finite at 0 if and only if $a = 1$, always has a light tail.

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- R is the amplitude of an isotropic α-stable distribution $(0 < \alpha < 2)$ on $\mathbb{R}^d$: $R = |\mathbf{Z}|$. Using result of Wolfe (1975) on unimodality of isotropic stable laws, it can be shown that $r^{1-d}h(r)$ is decreasing, bounded at the origin, and has a heavy tail: $r^{1-d}h(r) \sim cr^{-(d+\alpha)}$.

Simulation method 1

$$\mathbf{X} = R\mathbf{S},$$

where $R \geq 0$ has pdf $h(r) = (1/k_*)r^{d-1}g(r)$ and $\mathbf{S}$ is uniform on the contour $\mathbb{S}_*$.

Simulating R is easy for cases mentioned above.

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Simulating $\mathbf{S}$:
In dimension $d = 2$, ok: can accumulate arc length as you move around the contour and use the inverse of that to simulate.

In dimension $d > 2$: how to simulate from manifold uniformly w.r.t. surface $\mathbb{S}_*$? Approximate answer: triangulate the surface and to sample uniformly from the triangulation, choose one face proportional to the area of the face, then sample uniformly from the face.

Simulation method 2

$$\mathbf{X} = T\mathbf{B},$$

where $T \geq 0$ is derived from $g(r)$ and $\mathbf{B}$ is uniform on the unit ball $\mathbb{B}_*$.

Simulating $\mathbf{B}$:
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Simulating T?

Spherical cubature

To find the normalizing constant k_*, one has to evaluate an integral over the unit sphere $\mathbb{S} \subset \mathbb{R}^d$. Generally not possible to evaluate exactly, so will have to use multivariate numerical integration (cubature) on the $(d-1)$-dimensional manifold.

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Exact formulas for polynomials!
Stroud integration - works in dim. 3 to 16 for smooth functions
Adaptive spherical cubature - use polar parameterization of the unit sphere and then use existing adaptive cubature on that rectangular domain.

Outline

1 Generalized spherical distributions

2 Ratios of stable r.v.

Ratios of stable r.v.

In several applications, the ratio of independent stable r.v. arise:

$$X = Z_1/Z_0, \tag{3}$$

where $Z_i \sim S(\alpha_i, \beta_i, \gamma_i, \delta_i; k)$.

For example, the sample ACF function of heavy tailed data with tail $RV_{-\alpha}$, $1 < \alpha < 2$, the lag 1 sample covariance

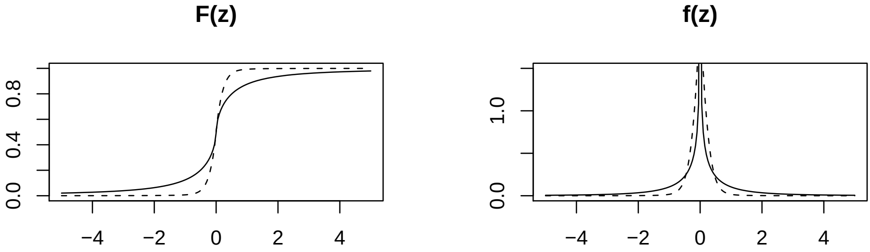
$$\frac{\sum_{i=1}^{n}(X_i X_i - \mu^2)}{\sum_{i=1}^{n} X_i^2}$$

converges when $n \to \infty$ to the ratio (3) with $\alpha_1 = \alpha$, $\alpha_0 = \alpha/2$.

Program to compute density, d.f. and quantiles of ratio

Standard formulas for density and d.f. of a ratio of independent r.v. are combined with numerical calculations of the densities/d.f. of the stable terms to get

- $F(t) = P(Z_1/Z_0 \leq t)$
- $f(t) = F'(t)$.
- quantile function $F^{-1}(u)$.



Distribution function and density of Z_1/Z_0, where $Z_1 \sim \mathbf{S}(\alpha, 0; 1)$ and $Z_0 \sim \mathbf{S}(\alpha/2, 1; 1)$. The solid line corresponds to $\alpha = 1.25$, the dashed line to $\alpha = 1.75$.

Tabulated critical values $F^{-1}(1 - p)$

For the ratio of symmetric α-stable and indep. positive $(\alpha/2)$-stable.

	p					
α	0.001	0.005	0.01	0.025	0.05	0.1
1.00	318.295	63.584	31.678	12.385	5.798	2.394
1.10	146.641	34.107	18.224	7.887	4.029	1.844
1.20	73.976	19.557	11.093	5.242	2.889	1.445
1.30	39.717	11.721	7.007	3.582	2.112	1.141
1.40	22.165	7.204	4.517	2.483	1.557	0.901
1.50	12.571	4.457	2.924	1.723	1.143	0.704
1.60	7.058	2.715	1.865	1.178	0.825	0.537
1.70	3.774	1.577	1.140	0.774	0.570	0.391
1.80	1.778	0.821	0.634	0.465	0.359	0.258
1.90	0.586	0.327	0.274	0.217	0.175	0.131
1.95	0.219	0.149	0.130	0.107	0.088	0.067