MURI Update Meeting
Multivariate Heavy Tail Phenomena: Modeling and Diagnostics
October 7, 2014,
Columbia University,

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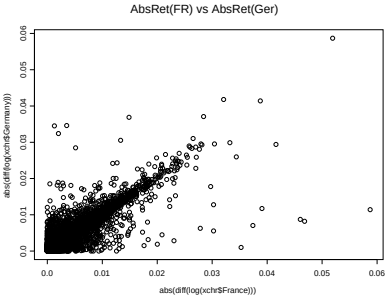
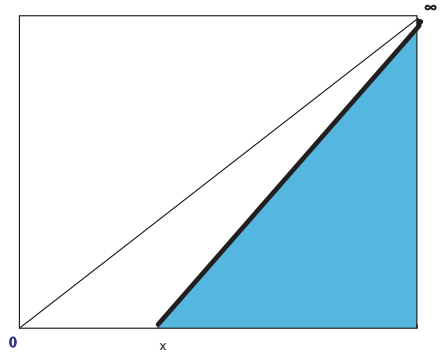
October 5, 2014

1. Mathematical framework

- Finished a survey giving proper mathematical framework in Lindskog, Resnick, and Roy (2014). Final submitted.
- Allows for flexible mathematical consideration of multivariate regular variation problems with multiple asymptotic regimes simultaneously existing.
- Ideal for handling risk contagion in presence of *asymptotic independence*. Also suitable for seeking multiple regular variation regimes in presence of *asymptotic full dependence*.



- Returns are multivariate heavy tailed.
- Limit measure concentrates on diagonal.
- Other examples
  - Xchr (Aus$, Chinese).
  - (Exxon, Chevron).
  - Tech sector returns? Lagged variables (Davis).



- If limit measure concentrates on the diagnonal, no mass in blue region

$$\{(u, v) : u > v + x\}.$$

- Wrong asymptotics?

Ongoing (Amy Willis, Bikram Das)

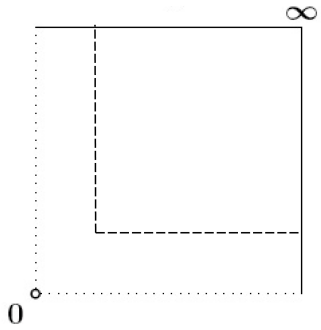
- Apply framework to cases other than asymptotic independence.
- Find data diagnostics.
- Dimension issues beyond 2.
- Discovery of regions where the limit measure has zero mass.

2. Model generation and diagnostic

2.1. A general construction of a standardized distribution

- On $\mathbb{R}^d_+$, delete closed cone F.
- Write
$$\aleph_F = \{\mathbf{x} : d(\mathbf{x}, F) = 1\}.$$
Take
$$\boldsymbol{\Theta} \text{ re in } \aleph_F, \quad R \sim \text{ Pareto}.$$
- Set
$$\boldsymbol{X} = R\boldsymbol{\Theta}$$
and $\boldsymbol{X} \in$ MRV on $R^d_+ \setminus F$.

- Can apply to multiple regimes by
  - first deleting F_1 (eg, origin)
  - then deleting $F_1 \cup F_2$; ie delete 2nd cone (eg, axes).



- How do the 2 regimes interact? Statistically identifiable? Das and Resnick (2014).

2.2. Model Detection Diagnostics

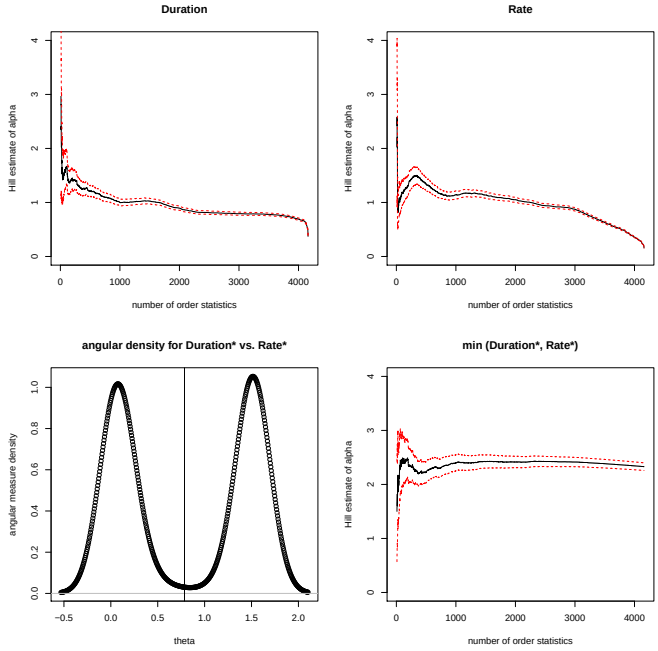
1. Reduction to one dimension:
   - $\boldsymbol{X} \in$ MRV iff $aX_1 \vee bX_2 \in RV(\alpha)$ for all $a \geq 0$, $b \geq 0$.
   - $\boldsymbol{X} \in$ HRV iff $aX_1 \wedge bX_2 \in RV(\alpha_0)$ for $a \wedge b > 0$.

   [Hint: Cannot check $\forall a, b$; works in higher dimensions.]
2. Use GPOLAR to convert to the CEV model and then use CEV diagnostics (Das and Resnick, 2011) using the *Hillish* and *Pickandsish* plots.
3. CEV model is of the form: Sppse $(\xi, \eta) \in \mathbb{R}_+ \times \mathbb{R}$ is a random vector and
   - $\exists b(t) \to \infty$,
   - $\exists$ limit measure μ

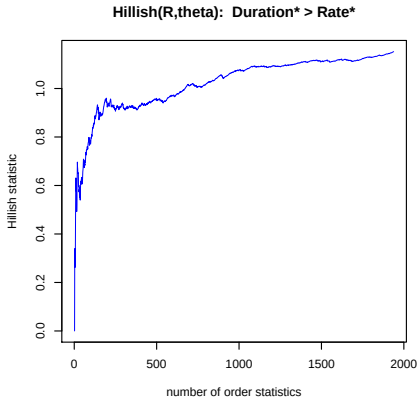
   such that
   $$tP\left[\left(\frac{\xi}{b(t)}, \eta\right) \in \cdot\right] \to \mu(\cdot).$$
   This is the form of convergence after coordinate transformation via GPOLAR.
4. Hillish statistic applied to both (ξ, η) and $(\xi, -\eta)$ detects this.

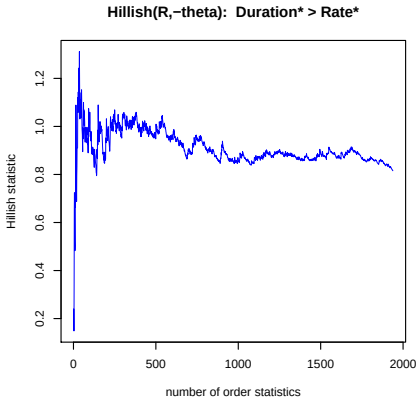
One dimensional analysis: BU (duration, rate).

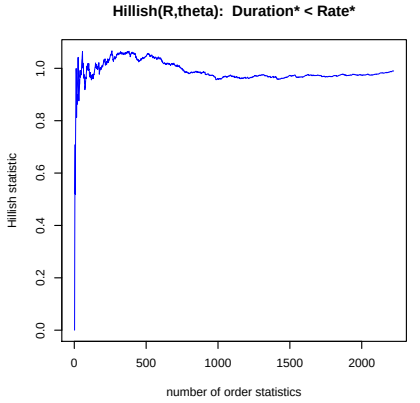
BU browser downloads; duration = duration of download; rate = file size/duration.

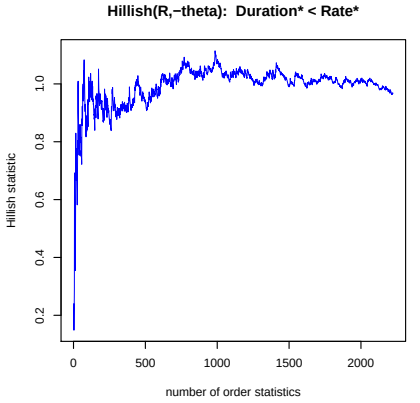


Hillish and CEV analysis for HRV.









Ongoing:

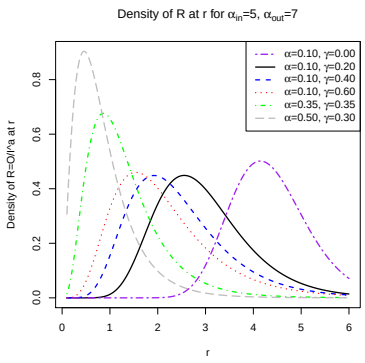
- Extensions to higher dimensions.
- Extensions to other deleted cones.
- More data examples.

3. Growing preferential attachment networks.

- Directed edge network grow according to preferential attachment postulates (Samorodnitsky, Resnick, Towsley, Davis, Willis, and Wan, 2014).
- Each node has (in-degree, out-degree) evolving with $n = 1, 2, \ldots$.
- At nth stage, the proportion of nodes with (in-degree, out-degree) $= (i, j)$ is

$$p_{ij}^{(n)} \overset{n\to\infty}{\to} p_{ij}.$$

- $\{p_{ij}\}$ is the mass function of a heavy tailed measure whose limit measure has a density that can be explicitly computed as a function of input parameters.
- General and powerful Tauberian theorem relating Laplace transform of (in-degree, out-degree) to the multivariate heavy tail (Resnick and Samorodnitsky, 2014).



- Algorithms (Atwood, Roy) to simulate the network growth.
- Provides test bed data for statistical methods where we know the answers (because we simulated the model). More from Joyjit.

Ongoing:

- Model error: Observe or infer frequencies $p_{ij}^{(n)}$ but try to infer something about heavy tailed structure of

$$\lim_{n\to\infty} p_{ij}^{(n)} = p_{ij}$$

  for $i \to \infty$ and $j \to \infty$.
- Theory for multivariate heavy tailed mass functions as opposed to their measures.
- Data is node based and not iid.
- Properties of estimators. Apply MG's and MG CLT.
- Simplify analysis using embedding in birth processes which are conditionally Poisson.
- Apply the Tauberian method to other models.
- Possibility of more than just 2 attributes per node (eg, friend, foe, strength of opinion).

- Other graceful methods to handle censoring.
- Neighborhood and community discovery (Davis, Wan, Zhang, Towsley, Jiang).

Contents

References

B. Das and S. Resnick. Generation and detection of multivariate regular variation and hidden regular variation. *ArXiv e-prints*, March 2014. URL http://adsabs.harvard.edu/abs/2014arXiv1403.5774D. Accepted pending revision in Stochastic Systems.

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F. Lindskog, S.I. Resnick, and J. Roy. Regularly varying measures on metric spaces: Hidden regular variation and hidden jumps. Technical report, School of ORIE, Cornell University, 2014. URL http://arxiv.org/abs/1307.5803. Accepted: Probability Surveys.

S.I. Resnick and G. Samorodnitsky. Tauberian Theory for Multivariate Regularly Varying Distributions with Application to Preferential Attachment Networks. *ArXiv e-prints*, June 2014. URL http://adsabs.harvard.edu/abs/2014arXiv1406.6395R.

G. Samorodnitsky, S. Resnick, D. Towsley, R. Davis, A. Willis, and P. Wan. Nonstandard regular variation of in-degree and out-degree in the preferential attachment model. *ArXiv e-prints*, May 2014. URL http://adsabs.harvard.edu/abs/2014arXiv1405.4882S.

A tale of tails!

Estimating power law indexes for network data

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October 07, 2014

Outline

Outline

Description of Datasets

Real-world network data

- Directed, signed social network data from technology news website `SLASHDOT`
- Dataset was provided to us by Zhi-li Zhang
- Conjectured to have "scale-free power-law like" behavior
- Main interest is to study multivariate tail behavior

Simulated data

- Preferential Attachment Model
- Yields directed network where asymptotically in-degrees and out-degrees exhibit power-law behavior with known tail-exponents
- Can be used to benchmark our methods
- We have developed a method to simulate from this model in $O(N)$ time where N is the number of nodes

Outline

Problems in tail-index estimation

- Usually power-law behavior is only asymptotic or may only be exhibited after a cutoff
- Forgoing the discrete assumption to fit continuous models lead to erros due to approximation
- Data might be also be censored as in the case of `SLASHDOT`

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- One popular package is `poweRlaw` package but the methods can be improved upon
- Slope based methods are inaccurate
- Other methods require tuning to yield credible estimates

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Our Goal

Find a statistical model for the data and an automated method to fit it that does not require manual threshold selection

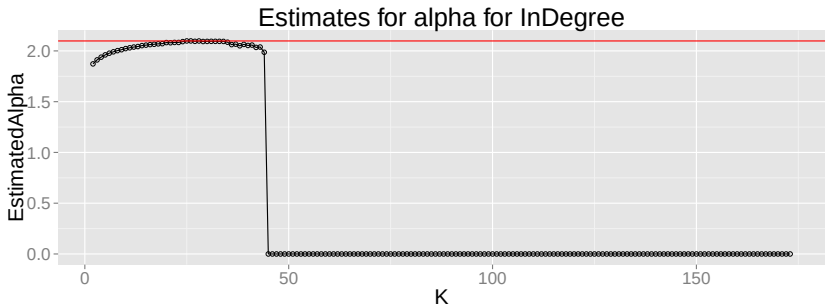
Outline

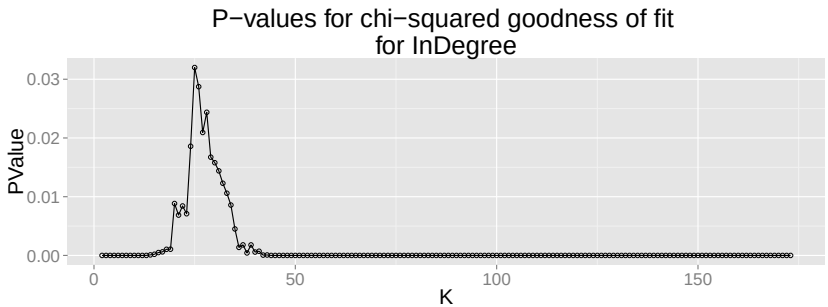
Using weighted linear regression

- Need to model dependence and the fact that the coefficients are constrained
- Let Y_i be the number of nodes with degree i. Assume
  - $Y_i \sim$ Multinomial(N, π) where N is the number of nodes
  - $\pi_i = Ci^{-\alpha}$ where α is the tail-index for i greater than threshold K
- Use the multivariate normal approximation to the multinomial and the delta method to obtain the approximate distribution for $\log Y_i$
- $\log Y_i \sim N\left(\log(1 - \sum_{i<K} \pi_i) - \log \zeta(K, \alpha) - \alpha \log i, \;\; \sigma_i(\alpha)\right)$ for $i \geq K$, where $\zeta(K, \alpha) = \sum_{i \geq K} i^{-\alpha}$
- Fit a model with parameters $(\pi_1, \ldots, \pi_{K-1}, \alpha)$ using iteratively reweighted least squares and choose K via a goodness of fit criterion
- Fast and reliable but iterative method to obtain estimate is unstable sometimes

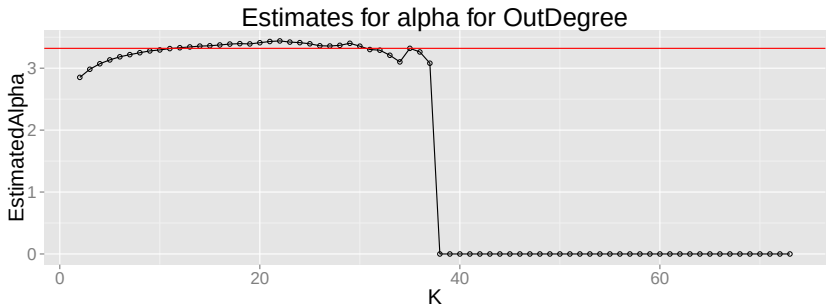
Comments

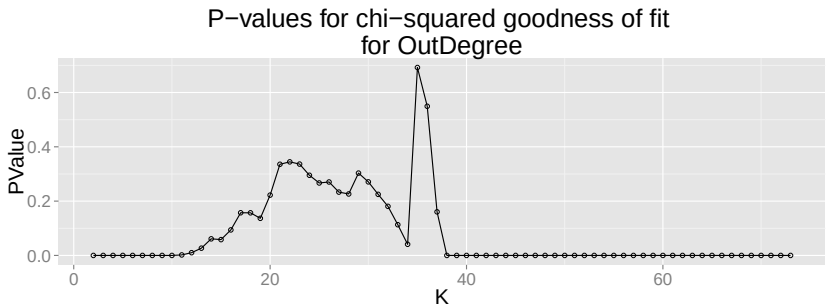
`SLASHDOT` data analysis of in-degree





Simulated data with tail-index of out-degree $= 3.67$





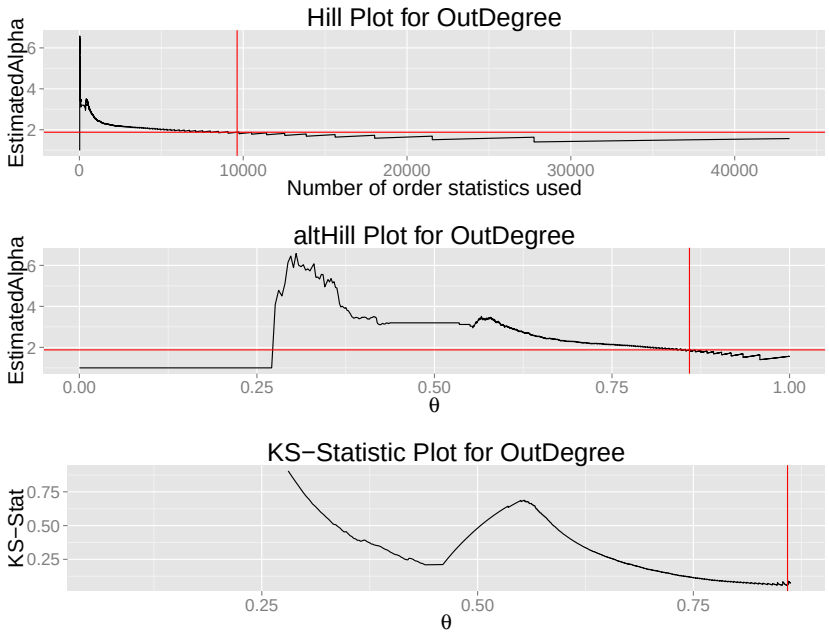
Outline

Using Likelihood based methods

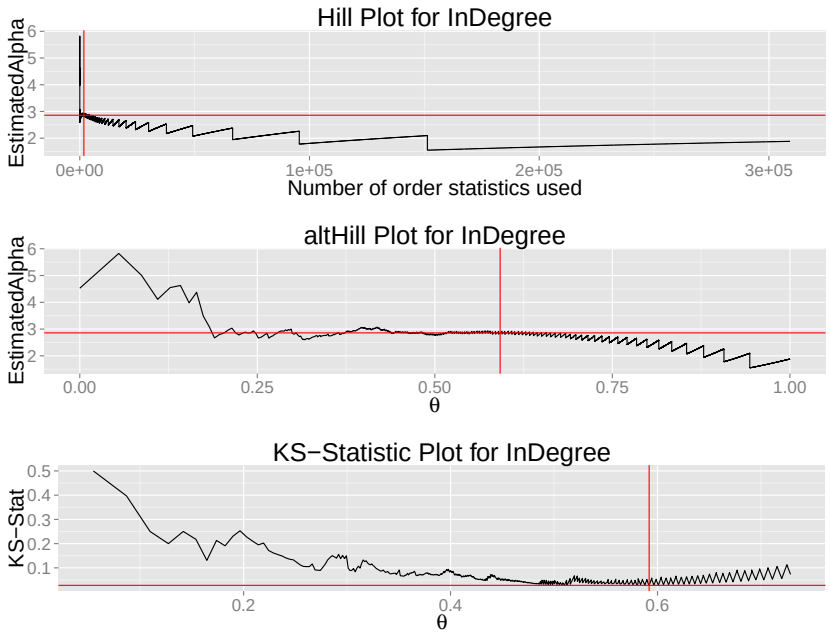
- Let $X_i, i = 1, 2, \ldots, N$ be the degrees of the N nodes in the graph. Assume X_i are iid and $\mathbb{P}[X = j | X > K] = \frac{j^{-\alpha}}{\zeta(K,\alpha)}$ for some threshold K. Recall $\zeta(K, \alpha) = \sum_{i \geq K} i^{-\alpha}$.
- Likelihood equation does not have a closed form solution for the maximum
- We also modify the standard likelihood to deal with censored data points
- For every K, we find a near optimal solution to the MLE, $\hat{\alpha}_K$, by carefully approximating the ζ-function
- Choose α_K which minimizes the Kolmogorov-Smirnov distance between the tail empirical distribution thresholded at K and our theoretical distribution
- This is nothing but a modified version of the Hill Estimator
- Method corresponds well to the traditional method of picking out approximate linear portion of Hill Plots
- Can iteratively improve estimate by improving approximation to ζ but the process is slow

Comments

SLASHDOT data analysis of out-degree



Simulated data with tail-index of in-degree = 3



Outline

Performance Comparison

- Both methods yield estimates which agree with each other
- When the true parameters are known, they outperform traditional methods
- Both methods yield ready made measures of uncertainty in estimation

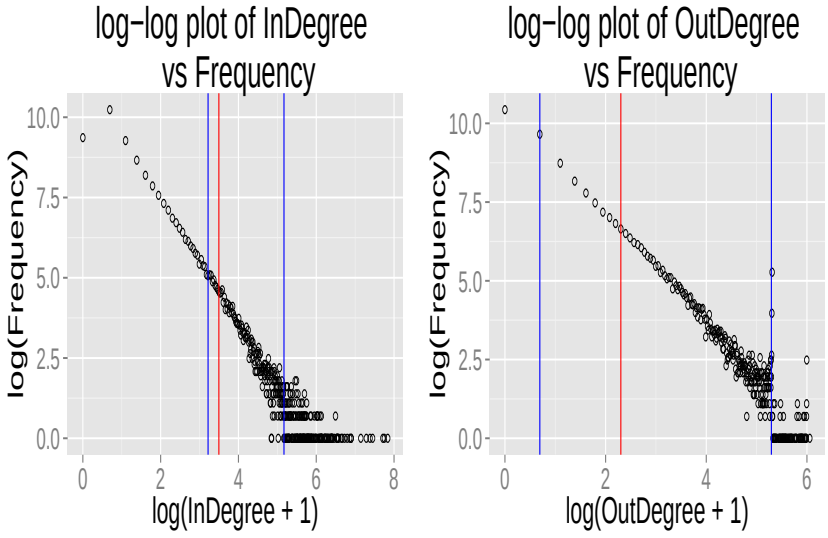
Outline

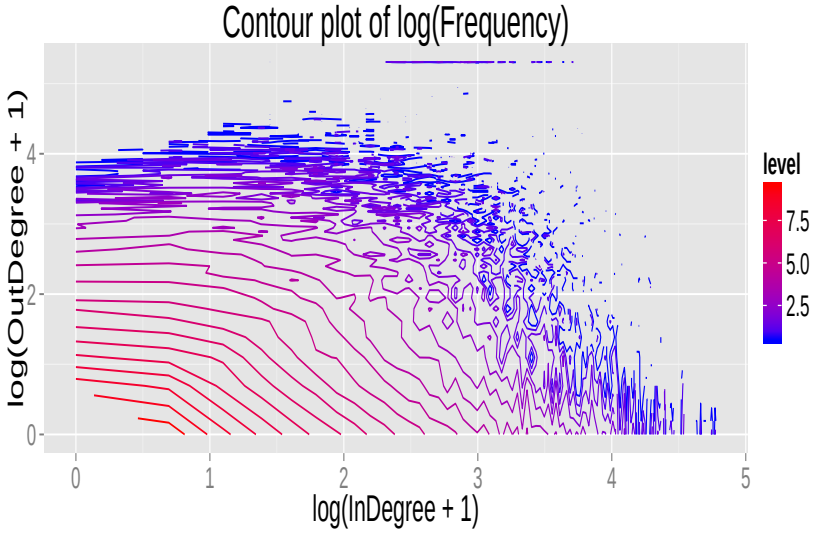
Towards Multivariate Methods

- No easily and reliably estimable statistical model is known
- Traditional methods only work when tail-indexes of both variables are the same which is rarely the case
- Extending the weighted regression method with terms to account for tail-dependence seem to work but still requires manual tuning
- Estimating measures like angular densities after standardizing marginals via rank-transforms also seem promising but again is not yet fully automated
- Both methods seem to be sensitive to model parameter selection
- Still qualitatively as well quantitatively, if we were to trust our not that robust multivariate methods, we can see distinct differences between the `SLASHDOT` dataset and simulated datasets from the model
- Simply cannot chalk-off these discrepancies to the fact that the `SLASHDOT` out-degrees are bounded

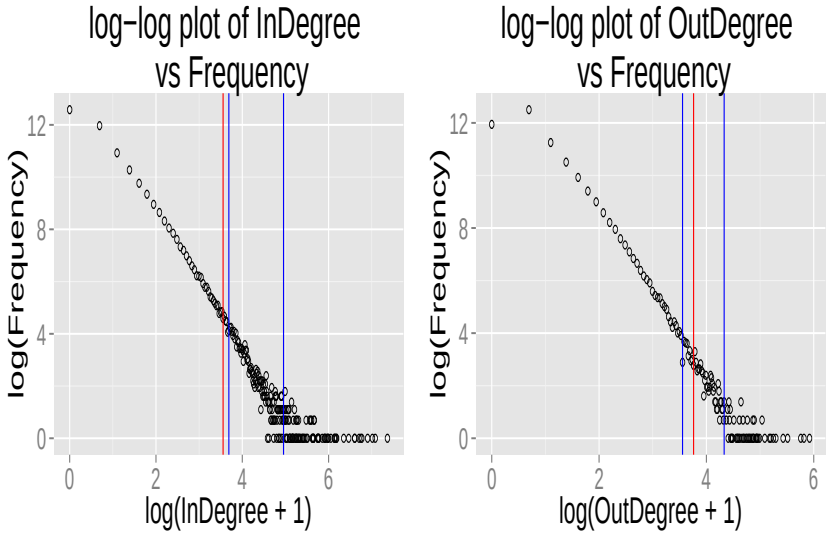
Caveat

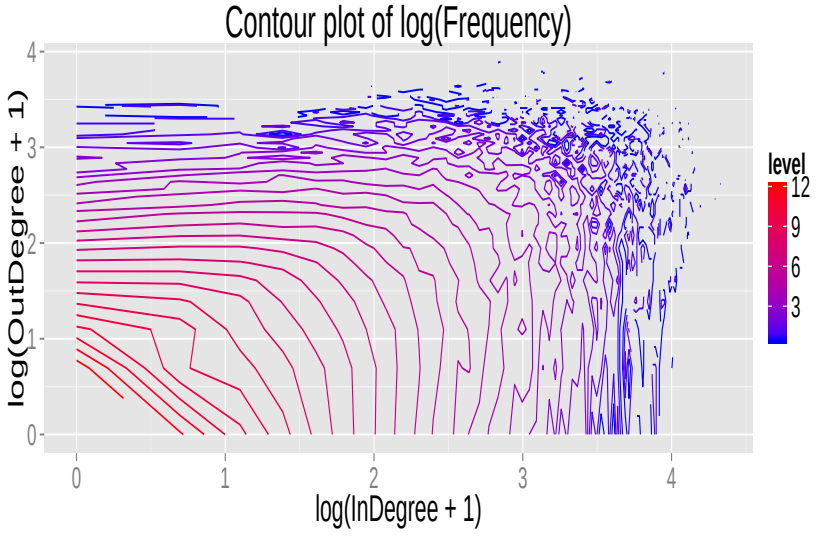
SLASHDOT data





Simulated data





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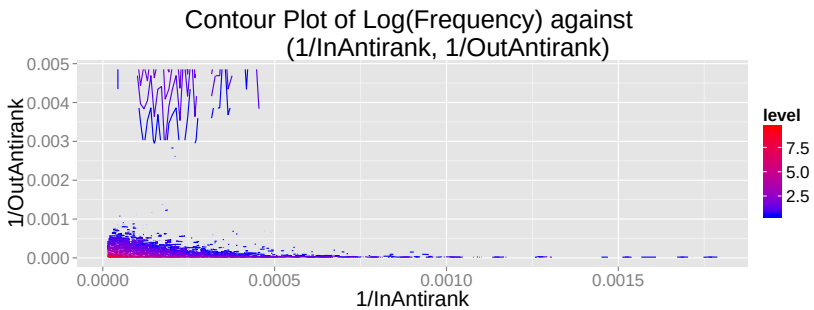

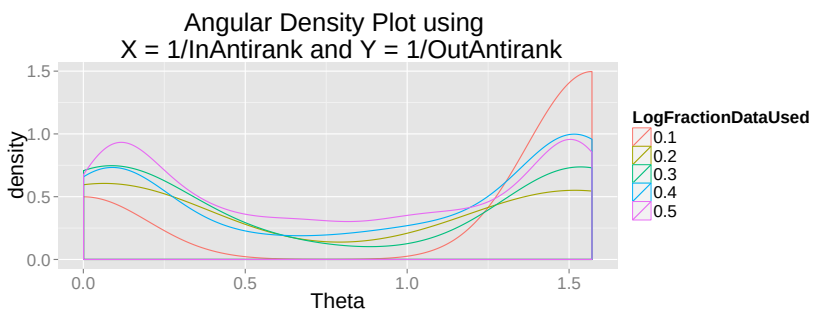

Simulated data