Update on Projects

Sidney Resnick
School of Operations Research and Information Engineering
Rhodes Hall, Cornell University
Ithaca NY 14853 USA

http://people.orie.cornell.edu/∼sid
sir1@cornell.edu

MURI CFEM Update

March 5, 2015

CORNELL

1. Undirected growing preferential attachment graphs

Notation:

- G_n=graph after n changes; node set is $V_n = \{1, \ldots, n\}$.
- $D_n(v)$= degree of $v \in V_n$.
- $\delta \geq -1$ = parameter.
- Initialize with one node having a self loop.

Conditional on knowing the graph G_n, at stage $n+1$ a new node $n+1$ appears and with a parameter $\delta \geq -1$, either

1. The new node $n+1$ attaches to $v \in V_n$ with probability

$$\frac{D_n(v)+\delta}{n(2+\delta)+(1+\delta)}, \tag{1}$$

or

2. $n+1$ attaches to itself with probability

$$\frac{1+\delta}{n(2+\delta)+(1+\delta)}. \tag{2}$$

In the first case

- $D_{n+1}(n+1) = 1$ and in the second case
- $D_{n+1}(n+1) = 2$.

Frequencies: The number of nodes at time n with degree k is

$$N_n(k) = \sum_{v=1}^{n} 1_{[D_n(v)=k]}$$

and

$$\frac{N_n(k)}{n} \to p_k, \quad (n \to \infty)$$

and

$$p_k = \Big(\frac{(2+\delta)\Gamma(3+2\delta)}{\Gamma(1+\delta)}\Big)\frac{\Gamma(k+\delta)}{\Gamma(k+3+2\delta)} = c(\delta)\frac{\Gamma(k+\delta)}{\Gamma(k+3+2\delta)}.$$

and

$$p_k \sim c(\delta)k^{-3-\delta}, \quad (k \to \infty).$$

1.1. CLT for counts

With Gena: Using MG CLT:

$$\sqrt{n}\Big(\frac{N_n(k)}{n} - p_k\Big) \Rightarrow N(0, \sigma_k^2(\delta))$$

and also jointly in k. Requires knowing order of

$$\Big|\frac{E(N_n(k))}{n} - p_k\Big|.$$

1.2. Tail estimation: Estimate δ.

MG convergence theorem: For each v:

$$\frac{D_n(v)}{n^{1/(2+\delta)}} \to \xi_v, \quad (n \to \infty).$$

Known

$$\xi_v \stackrel{d}{=} \xi_1 \prod_{j=1}^{v} B_i,$$

whre B_i are beta rv's.
Known

$$\bigvee_{v \in V_n} \frac{D_n(v)}{n^{1/(2+\delta)}} \to \bigvee_{v} \xi_v < \infty.$$

Should imply

$$\sum_{v \in V_n} \epsilon_{\frac{D_n(v)}{n^{1/(2+\delta)}}} \Rightarrow \sum_v \epsilon_{\xi_v}$$

in $M_+(0,\infty]$.

1.2.1. Goals:

With node based data:

- Not much is known about the ξ_v's. Is there any connection to Poisson processes. (Yes when $\delta = 0$. [Bollobas & Riordan])
- Connection to tail empirical measure? Use this to get convergence to mean measure.
- Imply the Hill estimator is consistent?

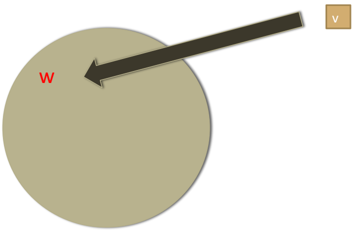
2. Directed edge preferential attachment

2.1. Model description

- Model parameters: $\alpha, \beta, \gamma, \delta_{\text{in}}, \delta_{\text{out}}$ with $\alpha + \beta + \gamma = 1$.
- $G(n)$ is a directed random graph with n edges, $N(n)$ nodes.
- Set of nodes of $G(n)$ is V_n; so $|V_n| = N(n)$.
- Set of edges of $G(n)$ is $E_n = \{(u,v) \in V_n \times V_n : (u,v) \in E_n\}$.
- In-degree of v is $D_{\text{in}}(v)$; out-degree of v is D_{out}. Dependence on n is suppressed.
- Obtain graph $G(n)$ from $G(n-1)$ in a Markovian way as follows:

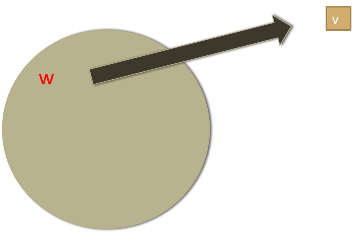
1. With probability α, append to $G(n-1)$ a new node $v \notin V_{n-1}$ and create directed edge $v \mapsto w \in V_{n-1}$ with probability

$$\frac{D_{\text{in}}(w) + \delta_{\text{in}}}{n - 1 + \delta_{\text{in}} N(n-1)}.$$



2. With probability γ, append to $G(n-1)$ a new node $v \notin V_{n-1}$ and create directed edge $w \in V_{n-1} \mapsto v \notin V_{n-1}$ with probability

$$\frac{D_{\text{out}}(w) + \delta_{\text{out}}}{n - 1 + \delta_{\text{out}} N(n-1)}.$$

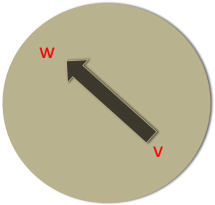


3. With probability β, create new directed edge between existing nodes

$$v \in V_{n-1} \mapsto w \in V_{n-1}$$

with probability

$$\Big(\frac{D_{\text{out}}(v) + \delta_{\text{out}}}{n - 1 + \delta_{\text{out}} N(n-1)}\Big)\Big(\frac{D_{\text{in}}(w) + \delta_{\text{in}}}{n - 1 + \delta_{\text{in}} N(n-1)}\Big).$$



Undirected
Directed
Extremes
Title Page
Page 7 of 14
Go Back
Full Screen
Close
Quit

2.2. CLT for count ratios.

With Tiandong Wang:
Let

$$N_n(k,l) = \sum_{v \in V_n} 1_{[D_{\text{in}}(v)=k, D_{\text{out}}(v)=l]}$$
$$= \sum_{v \in V_n} \#\text{ nodes with in-degree } k, \text{ out-degree } l.$$

Know

$$\frac{N_n(k,l)}{\#\text{ nodes at stage } n} \to p(k,l)$$

and $\{p(k,l)\}$ has a distribution which is regularly varying.

2.2.1. Goals:

- Use MG CLT to study asymptotic normality of

$$\sqrt{n}\Big(\frac{N_n(k,l)}{\#\text{ nodes at stage } n} - p(k,l)\Big)$$

- Eventually to find estimators for the parameters

$$\alpha_{\text{in}}, \alpha_{\text{out}}$$

and

$$\delta_{\mathrm{in}}, \delta_{out}, \alpha, \beta, \gamma.$$

2.3. Reciprocity

With Gena.

For the directed edge model,

> What proportion of nodes have reciprocated edges? Find an asymptotic behavior as the number of edges $\to \infty$. Express this asymptotic as a function of input parameters.

Markov chain approach: Let the set of edges be,

$$E_n = \{(u, v) : u \in V_n, v \in V_n, u \mapsto v \text{ or } v \mapsto u\}.$$

Define

$$\begin{aligned} S_n(u, v) = &0, \text{ if } (u, v) \notin E_n, (v, u) \notin E_n, \\ &1, \text{ if } (u, v) \in E_n, (v, u) \notin E_n, \\ &2, \text{ if } (u, v) \notin E_n, (v, u) \in E_n, \\ &3, \text{ if } (u, v) \in E_n, (v, u) \in E_n, \end{aligned}$$

3. rth largest

With Ross Maller, Boris Buchmann (ANU).
Setup:

- $\{X_n\}$ iid with a continuous df.
- $M_n^{(r)}$= rth largest among $X_1, \ldots, X_n$.
- $\boldsymbol{M}^{(r)} = \{M_n^{(r)}, n \geq r\} \in \mathbb{R}^\infty$.

  Fact: $\{\boldsymbol{M}^{(r)}, r \geq 1\}$ is a Markov chain on $\mathbb{R}^\infty$.
- Relative ranks:

$$R_n = \sum_{j=1}^{n} 1_{[X_j \geq X_n]}$$

$$=\text{relative rank of } X_n \text{ among } X_1, \ldots, X_n$$

$$=\text{rank of } X_n \text{ at "birth"}.$$

- Define the r-record times of $\{X_n\}$ by

$$L_0^{(r)} = 0, \quad L_{n+1}^{(r)} = \inf\{j > L_n^{(r)} : R_j = r\}$$

  and the r-records are $\{X_{L_n^{(r)}}, n \geq 1\}$ which are points of PRM $(R(dx))$ by Ignatov's theorem.

3.1. Facts:

- $\mathcal{R}_r$, the range of $\boldsymbol{M}^{(r)}$ is

$$\mathcal{R}_r := \bigcup_{p=1}^{r} \{X_{L_n^{(p)}}, n \geq 1\},$$

  By Ignatov's theorem, this is a sum of r independent PRM(R) processes and therefore the range of $\boldsymbol{M}^{(r)}$ is PRM(rR).

- $\mathcal{R}_r$, the range of $\boldsymbol{M}^{(r)}$, converges as a random closed set in the Fell topology to $\mathcal{R}$, the support of the measure R:

$$\mathcal{R}_r \Rightarrow \mathcal{R},$$

  as $r \to \infty$.

- $\boldsymbol{M}^{(r)}$ jumps at time k iff

$$R_k \in \{1, \dots, r\},$$

  so

$$\{[\boldsymbol{M}^{(r)} \text{ jumps at } k], k \geq r\}$$

  are independent events over k and

$$P[\boldsymbol{M}^{(r)} \text{ jumps at } k] = \frac{r}{k}.$$

3.1.1. Question

Is there anything corresponding to a stationary distribution in the Markov chain sense. Is there a non-trivial limit as $r \to \infty$ for $\boldsymbol{M}^{(r)}$?

Contents