ICA Model with Log-Concave Density Estimations

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The Model

$$X = A \cdot S$$

- d-dimensional response $X = (x_1, \cdots, x_d)^T$
- d-dimensional independent components $S = (S_1, \cdots, S_d)^T$
- Full rank $d \times d$ transformation matrix A
- $S = W \cdot X$ with unmixing matrix $W = (w_1, \cdots, w_d)^T = A^{-1}$
- The Independent Component Analysis (ICA) model of the distribution of X

$$P(B) = \prod_{j=1}^{d} P_j(w_j^T B), \quad \forall B \in \mathcal{B}_d$$

- The goal is to recover the unmixing matrix W and $S = W \cdot X$

A Strategy: Project to the Space of Log-Concave Densities

- P_d: space of d-dimensional distributions satisfying non-singularity conditions
- $\mathcal{F}_d$: space of d-dimensional log-concave densities
- Log-concave: exponential of piece-wise linear densities, normal, Laplace
- Not log-concave: t, stable, Pareto
- Projection $\Psi^*(P) : P_d \to \mathcal{F}_d$

$$\Psi^*(P) := \text{argmax}_{f \in \mathcal{F}_d} \int_{\mathbb{R}^d} \log(f)\, dP$$

Projection to $\mathcal{F}_d^{\text{ICA}}$

Define $\mathcal{F}_d^{\text{ICA}}$ to be

$$\left\{ f \in \mathcal{F}_d : f(x) = |\det W| \prod_{j=1}^{d} f_j(w_j^T x), f_1, \cdots, f_d \in \mathcal{F}_1 \right\}$$

Theorem (Samworth and Yuan (2012))

If distribution P has density $f(x) = |\det W| \prod_{j=1}^{d} f_j(w_j^T x)$, then $\Psi^*(P) = \Psi^{**}(P) := \text{argmax}_{f \in \mathcal{F}_d^{\text{ICA}}} \int_{\mathbb{R}^d} \log(f)\, dP$, and it equals to

$$f^{**}(x) = |\det W| \prod_{j=1}^{d} f_j^*(w_j^T x),$$

where $f_j^* = \Psi^*(f_j)$.

Estimation Procedure

- Start from an arbitrary initial value of W
- Step 1: Find log-concave projection $\hat{f}_j^*$ of the distribution of $w_j^T X$
- Step 2: With $\hat{f}_j^*$, update W to maximize the log-likelihood

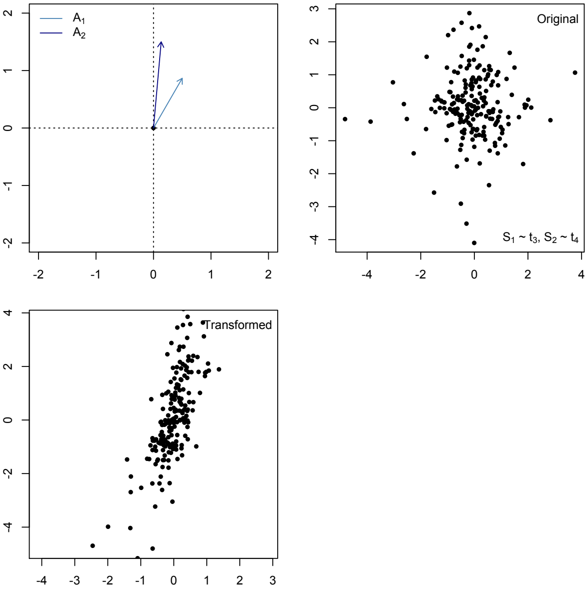
$$\log|\det W| + \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{d}\log \hat{f}_j^*(w_j^T x_i)$$

- Iterate steps 1 and 2, until convergence of the log-likelihood

Pre-Whitening

- Assume each component of S has finite variance (can relax, c.f. Chen and Bickel (2005))
- Let $\Sigma = \text{cov}(X)$ and $Z = \Sigma^{-1/2}X$
- $S = O \cdot Z$, where $O = W \cdot \Sigma^{-1/2}$ is an orthogonal matrix
- Number of unknown parameters is reduced from d^2 to $d(d-1)/2$

Non-Orthogonal Transformation, $S_1 \sim t_3$, $S_2 \sim t_4$



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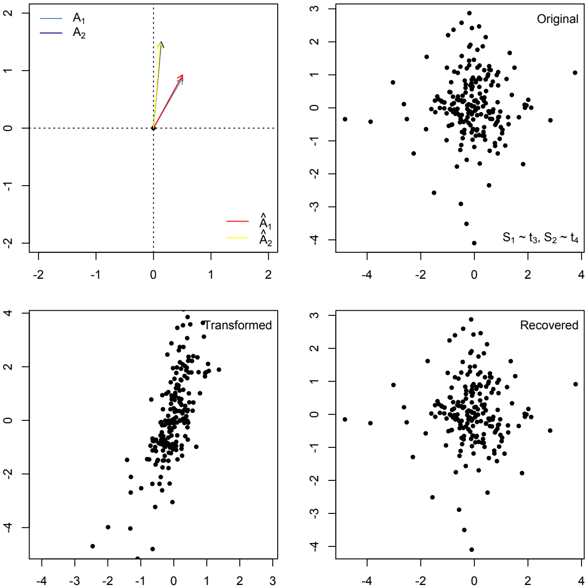
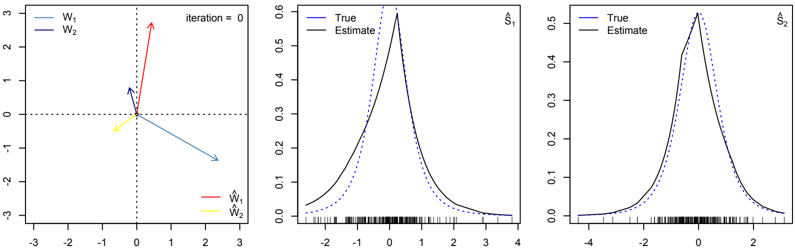


Convergence of Estimation

Non-orthogonal transformation, $S_1 \sim t_3$, $S_2 \sim t_4$



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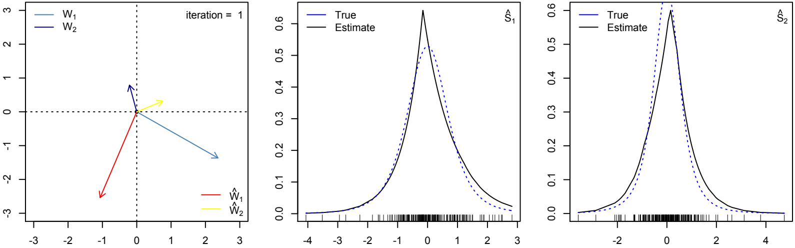

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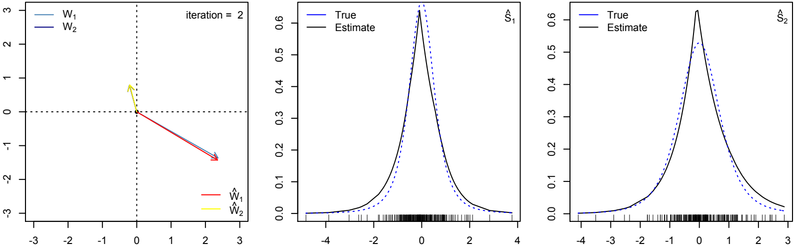
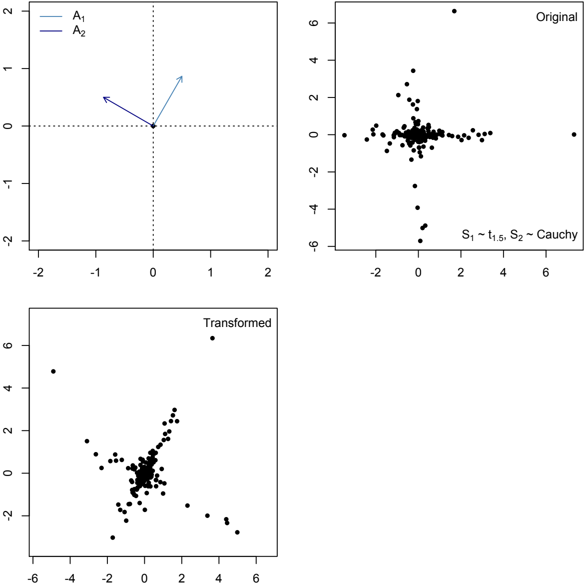


Rotation, $S_1 \sim t_{1.5}$, $S_2 \sim$ Cauchy



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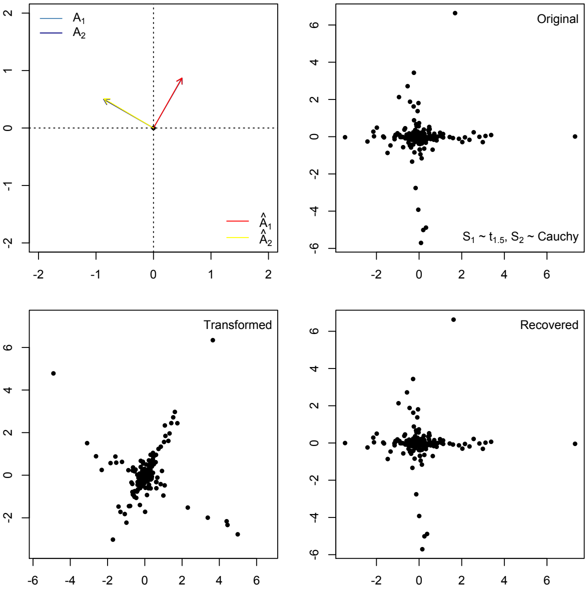
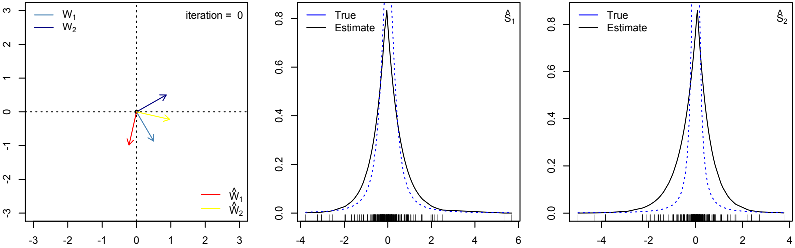


Convergence of Estimation

Non-orthogonal transformation, $S_1 \sim t_{1.5}$, $S_2 \sim \text{Cauchy}$



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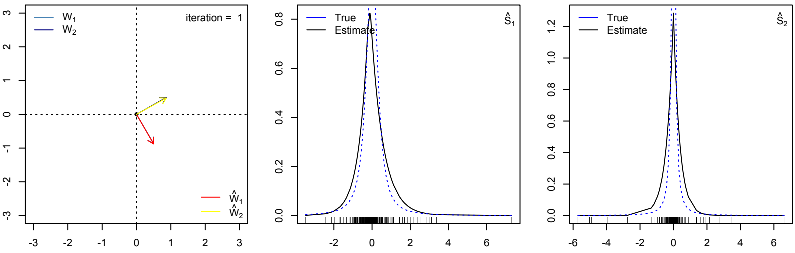

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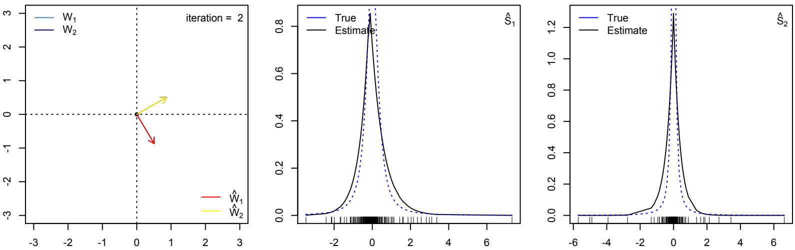


Overcomplete ICA

- d-dimensional response $X = (x_1, \cdots, x_d)^T$
- m-dimensional independent components $S = (S_1, \cdots, S_m)^T$
- $m > d$
- Full rank (non-degenerate) $d \times m$ transformation matrix $A = (a_1, \ldots, a_d)^T$
- $X = A \cdot S$
- The goal is to recover the transformation matrix A and the independent components S

Overcomplete ICA: Applications

- Estimate multivariate stable distributions
  - $S = (S_1, \cdots, S_m)^T$ and each S_j is univariate stable
  - $X = A \cdot S$ is multivariate stable
- Recognition tasks
  - Action recognition (Zhang et al., 2014)
  - Image feature extraction (Le et al., 2011)

Overcomplete ICA

- Recall the ICA model assumes the distribution of X when $m = d$ (undercomplete) is

$$P(B) = \prod_{j=1}^{m} P_j(w_j^T B), \quad \forall\, B \in \mathcal{B}_d,$$

  where $W = (w_1, \cdots, w_d)^T = A^{-1}$ exists when A is invertible
- Therefore for each W one can recover a unique estimate of S and compare with the proposed $\hat{P}_j$
- In particular, we use the log-concave projection $\hat{f}_j^*$ to estimate P_j and estimate W and $\hat{f}_j^*$ iteratively
- The difficulty in the overcomplete case is A is not invertible

Overcomplete ICA: Pre-Whitening

- Singular Value Decomposition (SVD) to reduce the number of parameters to estimate in the $d \times m$ matrix A
- $A = U\Sigma V^T$
- $U : d \times d$ orthogonal, $\Sigma : d \times d$ diagonal, $V : m \times d$ orthogonal
- As in the undercomplete case, assume each component S_j has finite variance (can relax by results in Chen and Bickel, 2005) and is standardized
- $\text{cov}(X) = U\Sigma^2 \tilde{V}^T$
- Let $Y = (U\Sigma)^{-1}X$, then $Y = V^T S$
- $(U\Sigma)^{-1}$ can be estimated using the SVD of the sample covariance of X

Pseudo-Inverse of the Transformation

- For pre-whitened under-complete ICA model $Y = V^T S$, where V is $d \times d$ orthogonal, $S = VY$, and V can be estimated by maximizing the log-likelihood function

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m}\log \hat{f}_j^*(s_{ji}) = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m}\log \hat{f}_j^*(v_j^T y_i)$$

- When V is $m \times d$ orthogonal matrix, $\{S : Y = V^T S\}$ is not unique
- A simple strategy is to use the pseudo-inverse of V, which is just V^T if V is orthogonal: $V^T V = I_d$
- Consistency may fail since $VV^T \neq I_m$ and thus $S \neq VY$

A Refined Strategy

- Solve for

$$\text{argmax}_{\hat{f},V}\left\{\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m}\log\hat{f}_j^*(s_{ji}^{\hat{f}^*}(V))\right\},$$

where

$$s^{\hat{f}}(V)=\text{argmax}_{\{s:Y=Vs\}}\left\{\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m}\log\hat{f}_j^*(s_{ji})\right\}$$

Estimation Procedure for Pre-Whitened Data

- Start from an arbitrary initial value of V and S such that $Y = VS$
- Step 1: Find log-concave projection $\hat{f}_j^*$ of the distribution of S_j
- Step 2: With $\hat{f}_j^*$, update V to maximize the log-likelihood

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m}\log \hat{f}_j^*(s_{ji}^{\hat{f}^*}(V)),$$

  which contains an optimization step

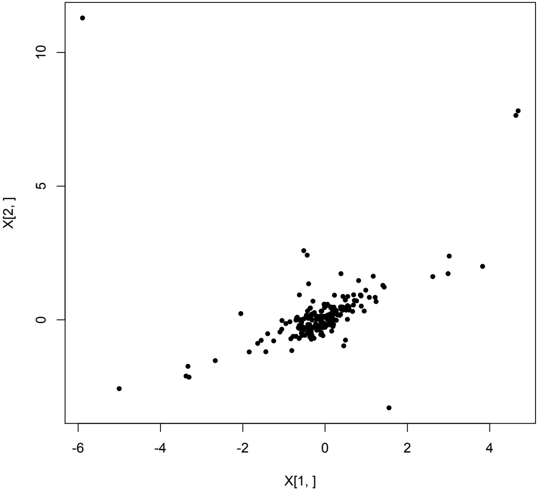
$$s^{\hat{f}}(V) = \text{argmax}_{\{s: Y=Vs\}}\left\{\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m}\log \hat{f}_j^*(s_{ji})\right\}$$

- Iterate steps 1 and 2, until convergence of the log-likelihood

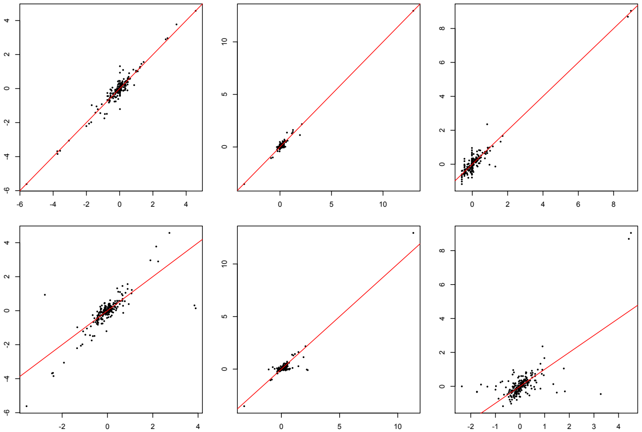
Example

- $m = 3, d = 2$
- S_1 : stable$(\alpha = 1.2, \beta = 0.1, \gamma = 1, \delta = 0)$
- S_2 : stable$(\alpha = 1.1, \beta = 0.7, \gamma = 1, \delta = 0)$
- S_3 : stable$(\alpha = 1.5, \beta = 0.3, \gamma = 1, \delta = 0)$
- Index parameter α; skewness β; scale γ; and location (shift) δ
- Transformation A: combinations of rotations with angles $(\pi/6, 2\pi/3, \pi/3)$

Data



Recovered S vs. Pseudo-Inverse



- y-axis: true values of S_j
- Top: recovered S given true V and log-concave projections $\hat{f}_j$
- Bottom: With pseudo-inverse $S = V^T Y$

References

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[2] Quoc V Le et al. "ICA with reconstruction cost for efficient overcomplete feature learning". In: (2011), pp. 1017–1025.

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[4] Shengping Zhang et al. "Action recognition based on overcomplete independent components analysis". In: *Information Sciences* 281 (Oct. 2014), pp. 635–647.