PCSDE Models for Bivariate Power-Law Behavior

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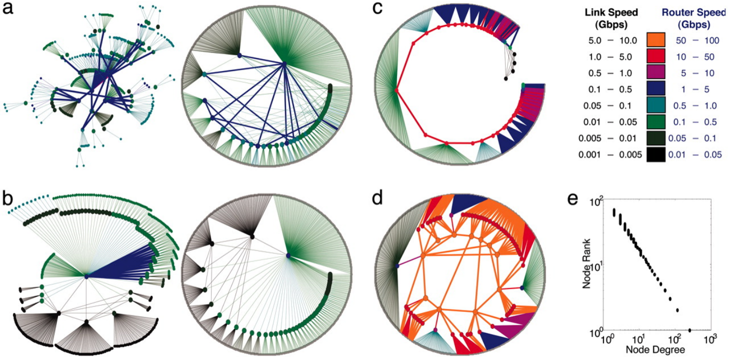
Overview

1. Past work
2. 1D PCSDE Model
3. 2D PCSDE Models
   - 2D PCSDE Model with a Shared Poisson Counter
   - Models with Markov on-off Modulation
   - Model with Coupled Differential Equations
4. 2D PCSDE Model with Brownian Motion (Ongoing work)
5. Conclusions

Past work

- Degree distribution is not enough - the false Achilles heel of the Internet
- Can one hear the shape of a complex network - spectral analysis in terms of the heat content function
- Computational methods for very large matrices in eigenvalue range intervals

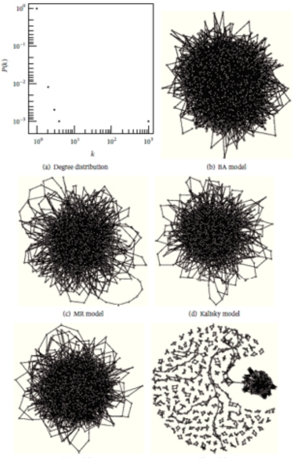
Degree distribution is not enough



Graphs with the same power law degree distribution

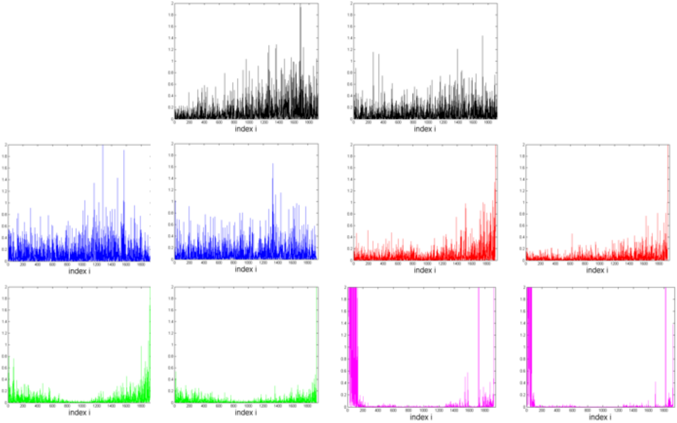
- Models for graphs with the same degree distribution
  - BA Model
  - Molly-Reed Model
  - Kalisky Model
  - Models of Grisi-Filho et al
    - Model A
    - Model B

Figure is from "*Scale-Free Networks with the Same Degree Distributions: Different Structural Properties*", *(J. H. H. Grisi-Filho, et al. 2013)*



(a) Degree distribution (b) BA model (c) MR model (d) Kalisky model (e) Model A (f) Model B

Alpha spectrum of power law graphs



Black: *B-A model;* ***Blue***: *M-R;* ***Red***: *Kalisky;* ***Green***: *Model A;* ***Magenta***: *Model B*

1D PCSDE Model for Power-Law Behaviors

Upper Tail Power-Law Generator

$$dX_t = \beta X_t dt + (x_0 - X_{t-})dN_t$$

- N_t is a Poisson counter with rate λ.
- Stationary density:

$$f_X(x) = \frac{\lambda}{\beta x_0}\left(\frac{x}{x_0}\right)^{-\frac{\lambda}{\beta}-1}, \quad x \geq x_0$$

- Complementary Cumulative Distribution Function (CCDF):

$$\bar{F}_X(x) = \left(\frac{x}{x_0}\right)^{-\frac{\lambda}{\beta}}, \quad x \geq x_0.$$

SDE Model with both Poisson Counter and Browian Motion

$$dX_t = \beta X_t dt + \sigma X_t dW_t + (x_0 - X_{t-}) dN_t$$

- Geometric Browian Motion (GBM) with Poisson Resetting.
- Stationay density: double-Pareto distribution [Reed, 2001].
- Power-law behavior in both tails.

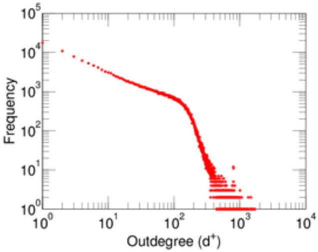
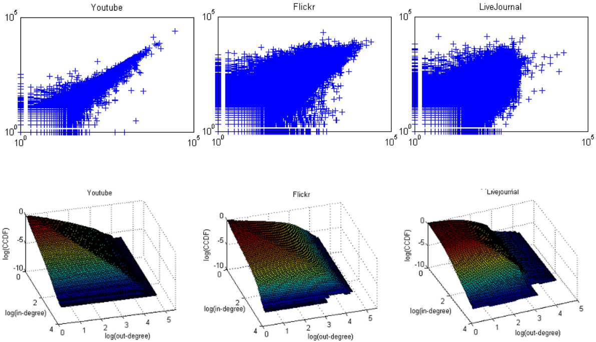


Figure: Twitter out-degree distribution

2D Power-Law in Real Data

- 2D Power-Law in Real Data [KONECT, 2013]:
  - Social Networks: Youtube, Flickr, Livejournal, etc.



- 2D PCSDE model as an explanation of correlated power law behavior in social networks?

2D PCSDE Model with a Shared Poisson Counter

Model formulation

$$dX_i = X_i dt + (1 - X_i)(dN_0 + dN_i), \;\; i = 1, 2$$

- N_0, N_1 and N_2 are independent Poisson counters with rates λ_0, λ_1 and λ_2. Let $\lambda_+ = \lambda_0 + \lambda_1 + \lambda_2$,

$$f_{X_i}(x_i) = (\lambda_0 + \lambda_i) x_i^{-(\lambda_0 + \lambda_i + 1)}, \;\; x_i \geq 1,$$

$$f_{X_1,X_2}(x_1, x_2) = \lambda_0 x_1^{-(\lambda_+ + 1)} \delta(x_1 - x_2) + \lambda_1 x_1^{-(\lambda_+ + 1)} f_{X_2}(x_2 x_1^{-1}) x_1^{-1} + \lambda_2 x_2^{-(\lambda_+ + 1)} f_{X_1}(x_1 x_2^{-1}) x_2^{-1}, \;\; x_1, x_2 \geq 1.$$

- Tail behavior: $\mathrm{P}(X_2 > x | X_1 > x) = \frac{\bar{F}_{X_1,X_2}(x,x)}{\bar{F}_{X_1}(x)} = x^{-\lambda_2} \xrightarrow{x \to \infty} 0.$

Markov on-off Modulation I

Model formulation

$$dX_i = X_i dt + (1 - X_i)\left((1 - Y)dN_0 + YdN_i\right), \; i = 1, 2$$

- Markov on-off Process Y_t,

$$dY_t = (1 - Y_t)dM_1 - Y_t dM_2, Y_0 \in \{0, 1\}$$

  M_1 and M_2 are independent Poisson counters with rates μ_1 and μ_2.
- The shared Poisson counter N_0 is effective when $Y_t = 0$;
- The independent Poisson counters N_1 and N_2 are effective when $Y_t = 1$.

- Use the characteristic function as in [JBGT, 2012];
- Marginal and Joint CCDF:

$$\bar{F}_{X_i}(x) = ax^{-A_i}b, \quad \bar{F}_{X_1,X_2}(x,x) = ax^{-A}b,$$

where

$$A_i = \begin{pmatrix} \lambda_0 & \lambda_i - \lambda_0 \\ -\mu_1 & \lambda_i + \mu_1 + \mu_2 \end{pmatrix}, \quad A = \begin{pmatrix} \lambda_0 & \sum_{i=1,2} \lambda_i - \lambda_0 \\ -\mu_1 & \sum_{i=1,2}(\lambda_i + \mu_i) \end{pmatrix},$$

with $a = (1,0)$, $b = (1, m(\infty))^T$ and $m(\infty) = \mathrm{E}[Y_\infty] = \frac{\mu_1}{\mu_1+\mu_2}$.

- Tail behavior: let $\xi^i_\pm$: eigenvalues of A_i; $\xi_\pm$: eigenvalues of A,

$$\xi^{(i)}_\pm = \frac{\lambda_0 + \lambda_i + \mu_1 + \mu_2}{2} \pm \frac{\sqrt{(\lambda_i - \lambda_0 + \mu_2 - \mu_1)^2 + 4\mu_1\mu_2}}{2}$$

$$\xi_\pm = \frac{\lambda_+ + \mu_1 + \mu_2}{2} \pm \frac{\sqrt{(\lambda_1 + \lambda_2 - \lambda_0 + \mu_2 - \mu_1)^2 + 4\mu_1\mu_2}}{2}.$$

  - Easy to check $\xi_- - \xi^{(1)}_- > 0$;
  - $\mathrm{P}(X_2 > x | X_1 > x) \sim C x^{-(\xi_- - \xi^{(1)}_-)} \xrightarrow{x \to \infty} 0$;
  - Still asymptotically independent.

Markov on-off Modulation II

Model formulation

$$dY = (1 - Y)dM_1 - YdM_2,$$
$$dX_i = X_i dt + (1 - X_i)((1 - Y)(dN_0 + dM_1) + Y(dN_i + dM_2)).$$

- X_i resets when Markov on-off process Y changes its state.
- Marginal and Joint CCDF:

$$\bar{F}_{X_i}(x) = ax^{-A_i}b \quad \bar{F}_{X_1,X_2}(x,x) = ax^{-A}b,$$

$$A_i = \begin{pmatrix} \lambda_0 + \mu_1 & \lambda_i + \mu_2 - \lambda_0 - \mu_1 \\ 0 & \lambda_i + \mu_2 \end{pmatrix},$$

$$A = \begin{pmatrix} \lambda_0 + \mu_1 & \sum_{i=1,2} \lambda_i + \mu_2 - \lambda_0 - \mu_1 \\ 0 & \sum_{i=1,2} \lambda_i + \mu_2 \end{pmatrix}.$$

- The spectral decomposition of A and A_i leads to

$$\bar{F}_{X_i}(x) = x^{-(\lambda_i+\mu_2)} m(\infty) + x^{-(\lambda_0+\mu_1)}(1 - m(\infty)),$$

$$\bar{F}_{X_1,X_2}(x,x) = x^{-(\lambda_1+\lambda_2+\mu_2)} m(\infty) + x^{-(\lambda_0+\mu_1)}(1 - m(\infty)).$$

- Tail behavior: let $\lambda_1 = \lambda_2 = \lambda$, $\lambda_0 + \mu_1 = \lambda_0'$, $\lambda + \mu_2 = \lambda'$,

$$\mathrm{P}(X_2 > x | X_1 > x) \xrightarrow{x \to \infty} \begin{cases} 1 & \lambda' > \lambda_0' \\ \frac{\mu_2}{\mu_1+\mu_2} & \lambda' = \lambda_0' \\ 0 & \lambda' < \lambda_0'. \end{cases}$$

  - Tail dependence coefficient goes to 1 or 0;
  - The case when tail dependence coefficient is fractional is not robust.

Interpretation of Modulation II

- Mixture of two models:

$$\begin{cases} dX_i = X_i dt + (1 - X_i) dN_0, & w.p.\ 1 - m(\infty); \\ dX_i = X_i dt + (1 - X_i) dN_i, & w.p.\ m(\infty), \end{cases}$$

  N_0 with rate λ_0' and $N_i,\ i = 1, 2$ with rate $\lambda_1 = \lambda_2 = \lambda'$.
- Tail behavior is determined by which model the observed large value more likely belongs to.
- A model with stable fractional tail dependence?

Model with Coupled Growth

Model formulation

$$d\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} dt + \begin{pmatrix} 1 - X_1 \\ 0 \end{pmatrix} dN_1 + \begin{pmatrix} 0 \\ 1 - X_2 \end{pmatrix} dN_2$$

- Marginal tail: let X_n be the value of $X_1(t)$ at the n^{th} arrival of the Poisson process N_2. Prove (X_n) satisfy a stochastic recursion

  $$X_{n+1} = A_{n+1}X_n + B_{n+1}, n = 1, 2, \ldots$$

  Then, for a stationary random variable X satisfy

  $$X \stackrel{d}{=} AX + B,$$

  we have $\mathrm{P}(X > x) \sim Cx^{-\alpha}, x \to \infty$. $\alpha > 0$ is such that $\mathrm{E}A^{\alpha} = 1$.

- Write the matirx

$$\underline{\underline{\beta}} = \begin{pmatrix} 1 & \beta \\ \beta & 1, \end{pmatrix}$$

$$\lambda_1 = \lambda_2 = \lambda.$$

- Note that the differential equation

$$d\underline{X}(t) = \underline{\underline{\beta}}\,\underline{X}(t)dt$$

has the solution

$$\begin{aligned} \underline{X}(t) &= e^{t\underline{\underline{\beta}}}\underline{X}(0) \\ &= \frac{1}{2}e^{t(1+\beta)}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\underline{X}(0) \\ &+ \frac{1}{2}e^{t(1-\beta)}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\underline{X}(0). \end{aligned} \tag{1}$$

- With (1), we compute A as follows.
- Let i.i.d. $(T_j) \sim \exp(2\lambda)$ independent of $N \sim Ge(1/2)$.

$$A = \begin{cases} \frac{e^{T_1(1+\beta)}+e^{T_1(1-\beta)}}{2} & N=0 \\ \frac{e^{T_1(1+\beta)}-e^{T_1(1-\beta)}}{2} \cdot \frac{e^{T_2(1+\beta)}-e^{T_2(1-\beta)}}{2} & \\ \cdot \prod_{j=3}^{N+1} \frac{e^{T_j(1+\beta)}+e^{T_j(1-\beta)}}{2} & N \geq 0 \end{cases} \tag{2}$$

- Solve α,

$$\mathrm{E}A^{\alpha} = \frac{1}{2} I_1 + I_2^2 \frac{1}{4-2I_1} = 1,$$

where

$$I_1 = \frac{\lambda 2^{-\alpha}}{\beta} \int_0^1 z^{\frac{2\lambda-\alpha(1+\beta)}{2\beta}-1} (1+z)^{\alpha} dz,$$

$$I_2 = \frac{\lambda 2^{-\alpha}}{\beta} B\left(\frac{2\lambda-\alpha(1+\beta)}{2\beta}, \alpha+1\right).$$

Numerical Results for Marginal Tail

- When $\beta = 0$, $\alpha = \lambda$.
- α decreases with β increasing.
- $\alpha > 0$ exists when $\mathrm{E}[\log A] < 0$.

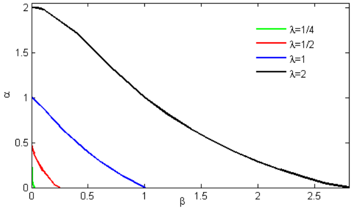


Figure: α as a function of β

- Let X be a random variable with the stationary distribution of the value of $X_1(t)$ at the moment when the counter N_2 has an arrival.
- Consider the combined counter $N_1 \cup N_2$. Its points are $W_1, W_2, \ldots$, with $(W_{n+1} - W_n)$ i.i.d, $\exp(2\lambda)$.
- The state of the system at these points has the stationary distribution

$$\begin{array}{ccc} \begin{pmatrix} 1 \\ X \end{pmatrix} & w.p. & \frac{1}{2} \\ \begin{pmatrix} X \\ 1 \end{pmatrix} & w.p. & \frac{1}{2} \end{array} \tag{3}$$

- Solution in (1) and (3) give the stationary distribution.

- Joint tail: let $T \sim \exp(2\lambda)$ and given $T = t$, $u \sim \mathbb{U}(0, t)$,

$$V = \frac{e^{u(1+\beta)} - e^{u(1-\beta)}}{2}; W = \frac{e^{u(1+\beta)} + e^{u(1-\beta)}}{2}.$$

  In the stationary regime,

$$(X_1, X_2) \overset{d}{=\!=} \begin{cases} (XV + W, XW + V) & w.p. \quad \frac{1}{2} \\ (XW + V, XV + W) & w.p. \quad \frac{1}{2} \end{cases}. \tag{4}$$

  where $\mathrm{P}(X > x) \sim Cx^{-\alpha}$.

- Tail behavior: with Breiman's lemma [Breiman, 1965],

$$\lim_{x\to\infty} \mathrm{P}(X_2 > x | X_1 > x) = \frac{2\mathrm{E}[V^\alpha]}{\mathrm{E}[V^\alpha] + \mathrm{E}[W^\alpha]},$$

Numerical results for Tail Dependence

- When $\beta = 0$, tail dependence coefficient equals 0.
- Tail dependent coefficient increases as β increases.
- Tail dependent coefficient approaches 1 when α approaches 0.

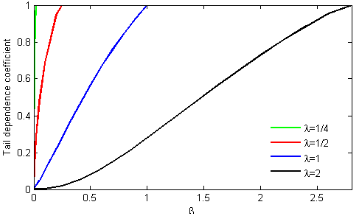


Figure: Tail dependence coefficient as a function of β

Generalization

Model formulation

$$d\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 1 & \beta_1 \\ \beta_2 & 1 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} dt + \begin{pmatrix} 1 - X_1 \\ 0 \end{pmatrix} dN_1 + \begin{pmatrix} 0 \\ 1 - X_2 \end{pmatrix} dN_2$$

- A is the same as in (2) with $\beta = \sqrt{\beta_1 \beta_2}$.
- Let $V_1 = \sqrt{\frac{\beta_1}{\beta_2}} V$ and $V_2 = \sqrt{\frac{\beta_2}{\beta_1}} V$,

$$\lim_{x \to \infty} \frac{\mathrm{P}(X_1 > x, X_2 > x)}{\mathrm{P}(X_i > x)} = \frac{\mathrm{E}\left[\min\left(W, V_1\right)^{\alpha}\right] + \mathrm{E}\left[\min\left(W, V_2\right)^{\alpha}\right]}{\mathrm{E}[V_i^{\alpha}] + \mathrm{E}[W^{\alpha}]}.$$

Numerical Results

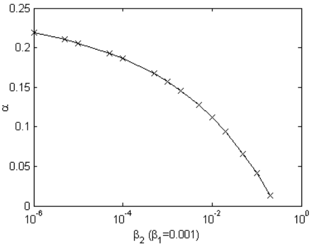


Figure: α as a function of β

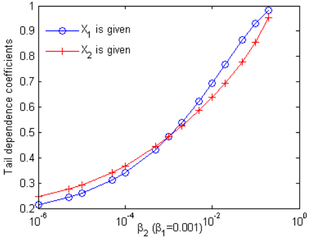
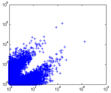


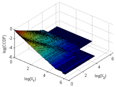
Figure: Tail dependence coefficient as a function of β

- Let $\lambda = 1/4$, fix $\beta_1 = 0.001$, the marginal tail α decreases with the increasing of β_2 value.
- The tail dependence coefficients with X_1 given or with X_2 given are different when $\beta_1 \neq \beta_2$.

For Real Data in Social Networks?

- The model with coupled differential equations has a feature not observed in 2D power-law data we know (but could be useful in modeling the prey-predator power law?) $(\lambda = 2, \beta = 0.2, \alpha = 1.9203.)$





- Go back to a single Poisson counter (two rare events occur together in the most likely way - the same cause)
- PCSDE model describe the expected degree growth.
- A Brownian motion component may help describing the randomness of degree growth.

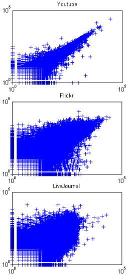


Figure: Real 2D data in Social Network

2D PCSDE Model with Brownian Motion I

Model Formulation

$$dX_1 = \beta_1 X_1 dt + \sigma_1 X_1 dW_1 + (1 - X_1) dN_0$$
$$dX_2 = \beta_2 X_2 dt + \sigma_2 X_2 dW_2 + (1 - X_2) dN_0$$

- Based on 1D Geometric Browian Motion with Poisson resetting.
- Let $Y_i = \log X_i$,

$$dY_i = \left(\beta_i - \frac{1}{2}\sigma_i^2\right) dt + \sigma dW_i - Y_i dN_0, \;\; i = 1, 2.$$

- Given $t \sim \exp(\lambda_0)$,

$$X_i(t) = \exp\left(\left(\beta_i - \frac{1}{2}\sigma_i^2\right) t + \sigma_i W_i(t)\right), \;\; i = 1, 2.$$

- Synthetic data generated by the model with Browian Motion I with with different σ values.
- The samples from this model do not fit well to the real data in social networks.

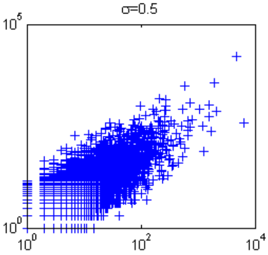


Figure: Synthetic data generated by the model with Brownian Motion I

How to modify the model to fit real data in social networks?

- With preferential attachment, each node is selected to be the target node with probability proportional to its current degree D.
- Think of dividing the node into D nodes with degree 1, each node will be selected as a target node with equal probability p.
- The new degree added to this node $d \sim \mathrm{B}(D, p)$ with mean Dp and variance $Dp(1-p)$.
- A reasonable approximation to $\mathrm{B}(D, p)$ when D when D is large is given by the normal distribution $\mathcal{N}(Dp, Dp(1-p))$.
- The variance is proportional to D. So, the standard deviation should be proportional to $\sqrt{D}$.

2D PCSDE Model with Brownian Motion II

Model Formulation

$$dX_1 = \beta_1 X_1 dt + \sigma_1 \sqrt{X_1} dW_1 + (1 - X_1) dN_0$$

$$dX_2 = \beta_2 X_2 dt + \sigma_2 \sqrt{X_2} dW_2 + (1 - X_2) dN_0$$

- Let $Y_i = \sqrt{X_i}$,

$$dY_i = \left(\frac{1}{2} \beta_i Y_i - \frac{1}{8} \sigma_i^2 \frac{1}{Y_i} \right) dt + \frac{1}{2} \sigma_i dW_i + (1 - Y_i) dN_0, \;\; i = 1, 2.$$

- For the tail, $Y_i \to \infty$,

$$dY_i = \frac{1}{2} \beta_i Y_i dt + \frac{1}{2} \sigma_i dW_i + (1 - Y_i) dN_0, \;\; i = 1, 2.$$

- Synthetic data:

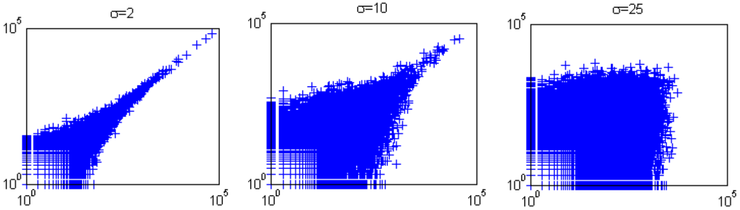
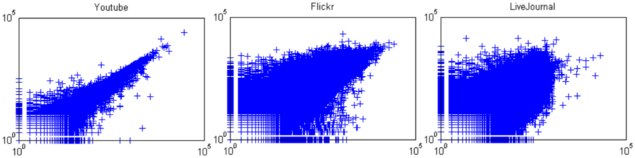


Figure: Synthetic data generated by the model with Brownian motion II

- Comparing to real data:



Other works

- David Mumford (1974 Fields medalist) in "Self-similarity of image statistics and image models": The hypothesis that natural images of the world, treated as a single large database, have renormalization invariant statistics has received remarkable confirmation from many quite distinct tests.
- Understanding the origin and generative mechanisms for scaling law in natural images is very important in developing more intelligent image processing methods.
- Our hypothesis is that human and animals have to be able to extract the same features against resolution blurring for survival. Mathematical study of this could provide new techniques in addition to the Scale Invariant Feature Transform (SIFT).

Conclusions

- We present a modulated sharing Poisson counter model with tail dependence coefficient to be 0 or 1. By adding a Brownian motion component to this model, we generate samples distributed like the ones observed in social networks.
- We also propose a model with fractional tail dependence coefficient. This model is interesting theoretically; however, the distribution of the samples generated by this model do not fit to the real data we know.

References

W.J. Reed (2001)
The Pareto, Zipf and other power-laws
Economics Letters 74, 15 – 19.

J. Kunegis (2013)
KONECT: the Koblenz Network Collection
WWW 2013 Companion May 13 – 17, 2013, Rio de Janeiro, Brazil.

B. Jiang, R.W. Brockett, W. Gong and D. Towsley (2012)
Stochastic Differential Eqations for power-law Behavior
1st IEEE Conference on Decision and Control Dec 10-13, 6696 – 6701.

L. Breiman (1965)
On Some Limit Theorems Similar to the Arc-Sin Law
Theory Probab. Appl. 10, 323 – 331.

Breiman's Theorem

Suppose that X and Y are two independent nonnegative random variables such that $\mathrm{P}(X > x)$ is regularly varying of index $-\alpha, \ \alpha \geq 0$, and that $\mathrm{E}[Y^{\alpha+\epsilon}] < \infty$ for some $\epsilon > 0$. Then

$$\mathrm{P}(XY > x) \sim \mathrm{E}[Y^{\alpha}]\mathrm{P}(X > x)$$

Thank You!