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Competitions in Nonlinear Pólya Urn Processes with Fitness

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MURI, April 15, 2016

Introduction

- Many social phenomena modeled as competition
  - e.g. online social tagging
- Important factors affecting competitions
  - Cumulative advantage: positive feedback, “rich get richer”
  - Fitness: intrinsic competitiveness
- Simplest model: Pólya urn
  - CA feedback linear

How do nonlinear CA & fitness interact?

Model

Nonlinear Pólya urn process with fitness

- two colors (1 and 2), add one ball at a time
- color k has $X_k(t)$ balls at time t
- color k has fitness f_k
- CA feedback strength $\beta \geq 0$

$$\mathbb{P}[\text{ball added at time } t+1 \text{ has color } k] = \frac{f_k X_k(t)^\beta}{f_1 X_1(t)^\beta + f_2 X_2(t)^\beta}$$

- depends on fitness only through ratio $r = f_1/f_2$
  - assume $r \geq 1$ by symmetry

Metrics

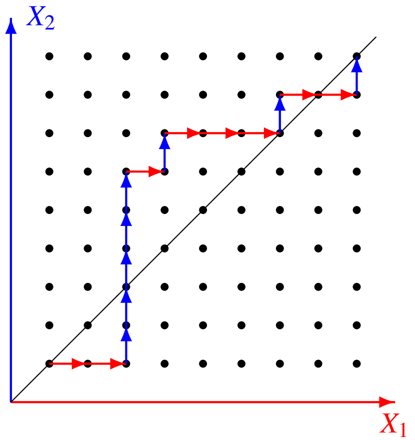
Given 2D process
$\{(X_1(t), X_2(t)) : t = 0, 1, 2 \ldots\}$

- Duration: time of last tie

$$T = \sup\{t \geq 0 : X_1(t) = X_2(t)\}$$

- Intensity: number of ties

$$N = \sum_{t=0}^{\infty} \mathbf{1}\{X_1(t) = X_2(t)\}$$



Stochastic Order: $r = 1$

Theorem

For equal fitness case ($r = 1$), stronger feedback (larger β) results in stochastically shorter and less intense competitions.

For $r = 1$, $\beta \geq \beta'$, same initial condition,

- $$\mathbb{P}[T \geq t \mid \beta] \leq \mathbb{P}[T \geq t \mid \beta'], \quad \forall t$$
- $$\mathbb{P}[N \geq n \mid \beta] \leq \mathbb{P}[N \geq n \mid \beta'], \quad \forall n$$

Proof.

By coupling argument. □

Stochastic Order: $r > 1$

Theorem

Feedback does not increase competition intensity.

$$\mathbb{P}[N \geq n \mid \beta \geq 0, r] \leq \mathbb{P}[N \geq n \mid \beta = 0, r]$$

Proof.

Again by coupling argument. □

Corollary

For $r > 1$, competition always ends, i.e. $T, N < \infty$ a.s..

Does Fittest Always Win?

- Yes, if $\beta \leq 1$
  - $\beta = 0, 1$: previously known
  - $\beta < 1$
- No, if $\beta > 1$
  - less fit can become monopoly (previously known)

Duration Distribution: $r = 1$

Theorem (Duration, equal fitness)

For $r = 1$,

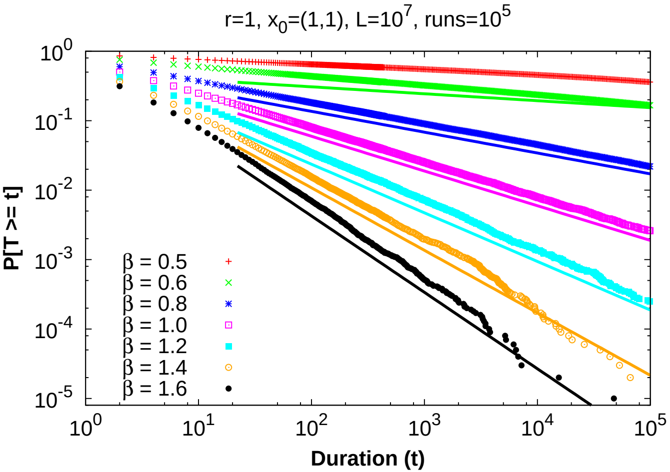
- *if $\beta \leq 1/2$, $T = \infty$ a.s. (previously known);*
- *if $\beta > 1/2$,*

$$\mathbb{P}[T \geq t] \sim Ct^{1/2-\beta}.$$

Proof.

Use exponential embedding and invariance principle. □

Duration Distribution: $r = 1$



Duration Distribution: $r > 1$

Theorem (Duration, different fitnesses)

For $r > 1$,

- *if $\beta > 1$,*

$$\mathbb{P}[T \geq t] \sim C_1 t^{1-\beta};$$

- *if $\beta = 1$,*

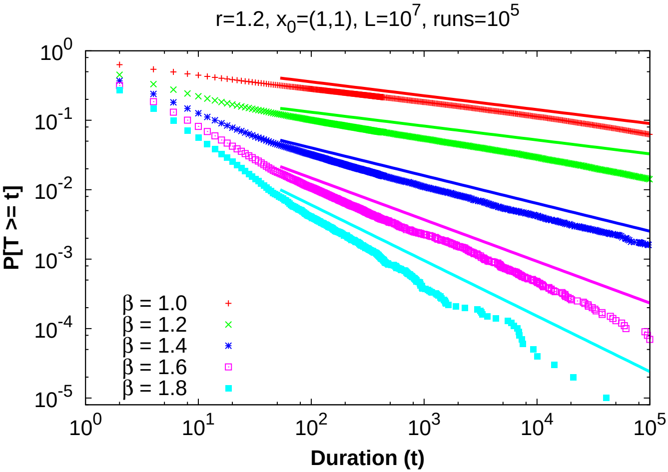
$$\mathbb{P}[T \geq t] = \Omega\left(t^{(1-r)x_{01}}\right) \cap O\left(t^{(1-r)(x_{01}-r^{-1})}\right);$$

- *if $\beta < 1$,*

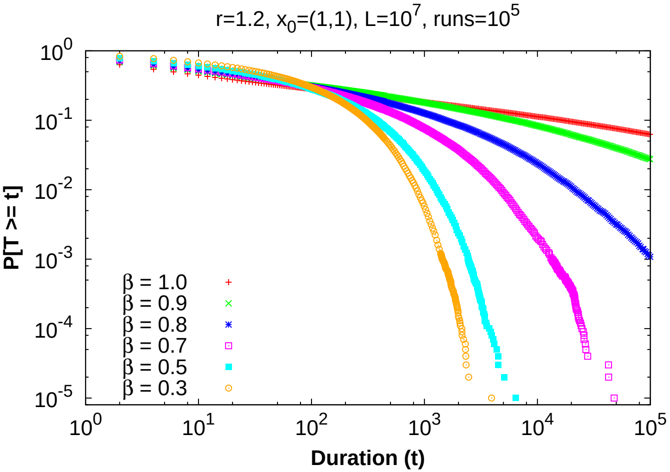
$$\mathbb{P}[T \geq t] = O\left(e^{-C_2 t^{1-\beta}}\right).$$

Proof.
Use exponential embedding, stochastic ordering & RW. □

Duration Distribution: $r > 1, \beta > 1$



Duration Distribution: $r > 1, \beta < 1$



Intensity Distribution: $r = 1$

Theorem (Intensity, equal fitness)

For $r = 1$,

- *if $\beta \leq 1/2$, $N = \infty$ a.s. (previously known);*
- *if $\beta \in (1/2, 1]$,*

$$\mathbb{P}[N \geq n] = \Omega(n^{-\beta}) \cap O(n^{1/2-\beta});$$

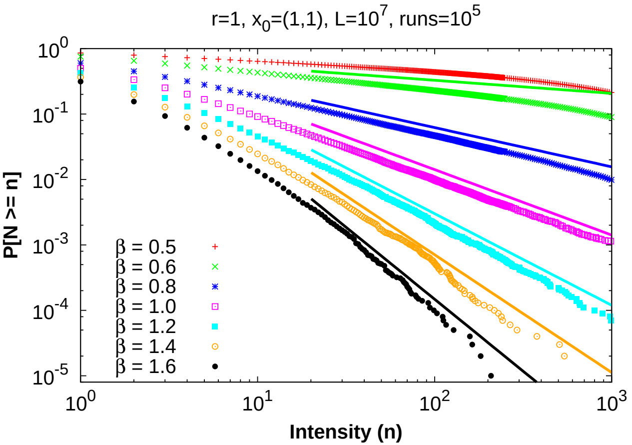
- *if $\beta \geq 1$,*

$$\mathbb{P}[N \geq n] = O(n^{-\beta}).$$

Proof.

Use convexity/concavity & RW. □

Intensity Distribution: $r = 1$



Conjecture: $\mathbb{P}[N \geq n] \sim C_3 n^{1-2\beta}$.

Intensity Distribution: $r > 1$

Theorem (Intensity, different fitness)

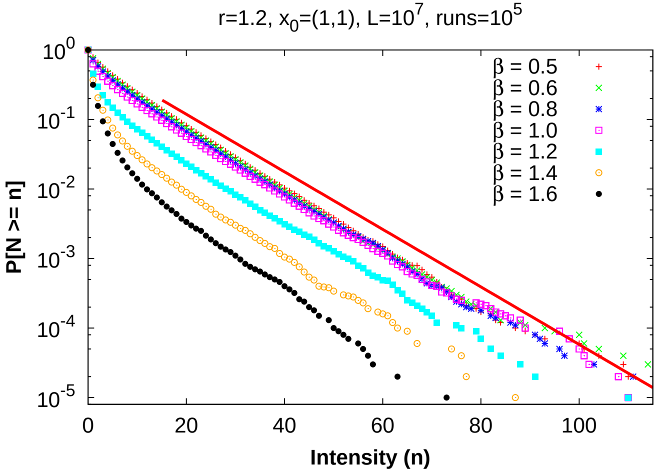
For $r > 1$ and all β,

$$\log \mathbb{P}[N \geq n] \sim n \log \frac{2}{r+1}.$$

Proof.

Relate to RW. □

Intensity Distribution: $r > 1$



Summary

Tail distributions of duration and intensity. Here $a = 2/(r+1)$.

	$\mathbb{P}[T \geq t]$		$\mathbb{P}[N \geq n]$	
	$r = 1$	$r > 1$	$r = 1$	$r > 1$
$0 \leq \beta \leq \frac{1}{2}$	1	$e^{-\Omega\left(t^{1-\beta}\right)}$	1	$O(a^n)$
$\frac{1}{2} < \beta < 1$	$\Theta(t^{\frac{1}{2}-\beta})$	$e^{-\Omega\left(t^{1-\beta}\right)}$	$\Omega(n^{-\beta})$	$O(a^n)$
$\beta = 1$	$\Theta(t^{-\frac{1}{2}})$	$\Omega(t^{(1-r)x_{01}})$	$\Theta(n^{-1})$	$O(a^n)$
$\beta > 1$	$\Theta(t^{\frac{1}{2}-\beta})$	$\Theta(t^{1-\beta})$	$O(n^{-\beta})$	$O(a^n)$

Future Work

- Prove conjecture
- More competitors
- Applications