A Poisson driven stochastic differential equation model with regularly varying tails

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April 15, 2016

Introduction

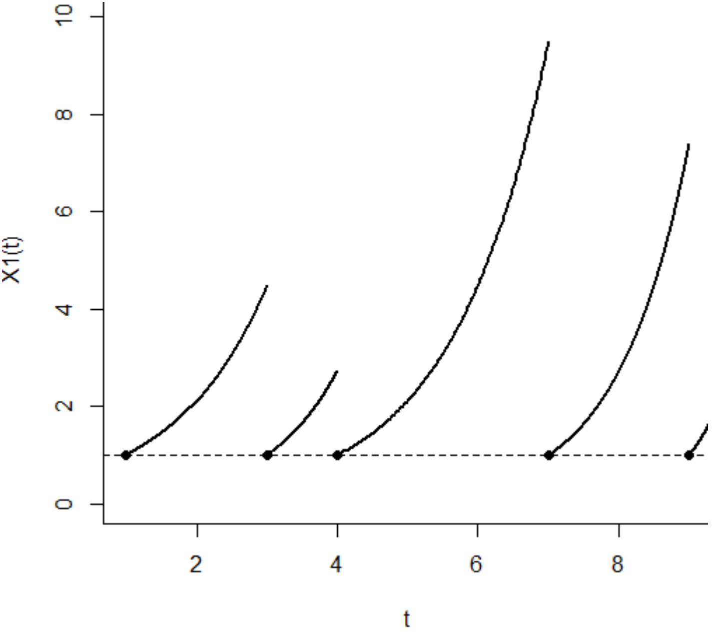
- Interest in power law behavior/regular variation in stochastic differential equation models
- We look at two-dimensional model that grows exponentially fast, but is reset at arrivals of two independent Poisson processes
- Goal: Prove converges to stationary distribution, find stationary distribution and show that it is regularly varying.

Model

General case:

- Independent Poisson processes N_1, N_2, with rates λ_1, λ_2
- State space $[1, \infty) \times [1, \infty)$
- Two dependent process, X_1, X_2, growing deterministically, reset to 1 at arrivals of N_1, N_2

$$dX_1(t) = \mu_{11} X_1(t)\,dt + \mu_{12} X_2(t)\,dt + (1 - X_1(t^-))\,dN_1(t)$$
$$dX_2(t) = \mu_{21} X_1(t)\,dt + \mu_{22} X_2(t)\,dt + (1 - X_2(t^-))\,dN_2(t)$$


X1(t)
0
2
4
6
8
10
2
4
6
8
t

Model

Symmetric case:

- Independent Poisson processes N_1, N_2, with rates $\lambda_1 = \lambda_2 := \lambda$
- State space $[1, \infty) \times [1, \infty)$
- Two dependent process, X_1, X_2, growing deterministically, reset to 1 at arrivals of N_1, N_2

$$
\begin{aligned}
dX_1(t) &= \quad X_1(t)\,dt + \mu X_2(t)\,dt + (1 - X_1(t^-))\,dN_1(t) \\
dX_2(t) &= \mu X_1(t)\,dt + \quad X_2(t)\,dt + (1 - X_2(t^-))\,dN_2(t).
\end{aligned}
$$

Model

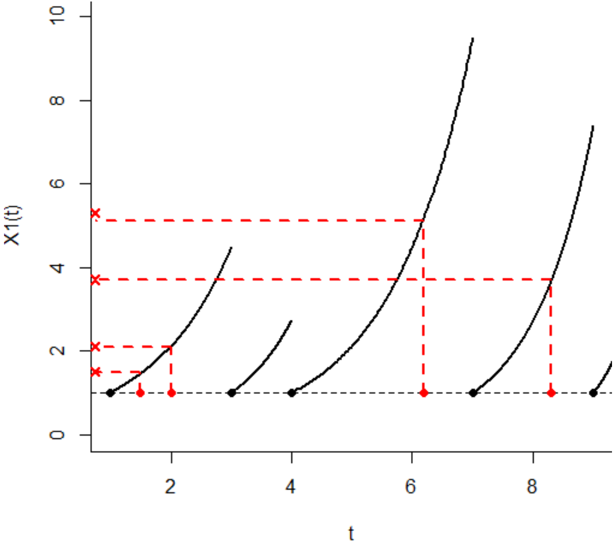
If no resets in $[0, t]$, deterministic growth according to

$$X_1(t) = \frac{e^{t(1+\mu)} + e^{t(1-\mu)}}{2} X_1(0) + \frac{e^{t(1+\mu)} - e^{t(1-\mu)}}{2} X_2(0)$$

$$X_2(t) = \frac{e^{t(1+\mu)} - e^{t(1-\mu)}}{2} X_1(0) + \frac{e^{t(1+\mu)} + e^{t(1-\mu)}}{2} X_2(0)$$

Sampled Process

Let's consider a convenient one-dimensional sampling...
Let X_n be the value of $X_1(t)$ at the n^{th} arrival of N_2



Joint Process

$Y \sim$ stationary distribution of (X_n).
At any arrival of $N_1 \cup N_2$, $(X_1(t), X_2(t))$ has stationary distribution

$$\begin{cases} (1, Y) & \text{w.p. } \frac{1}{2} \\ (Y, 1) & \text{w.p. } \frac{1}{2}, \end{cases}$$

and is deterministic in between!
Recall, if no arrivals in $[0, t]$,

$$X_1(t) = \frac{e^{t(1+\mu)} + e^{t(1-\mu)}}{2} X_1(0) + \frac{e^{t(1+\mu)} - e^{t(1-\mu)}}{2} X_2(0)$$
$$X_2(t) = \frac{e^{t(1+\mu)} - e^{t(1-\mu)}}{2} X_1(0) + \frac{e^{t(1+\mu)} + e^{t(1-\mu)}}{2} X_2(0)$$

Joint Process

So the continuous time stationary distribution looks like

$$(X_1, X_2) \stackrel{d}{=} \begin{cases} (\,YV + W, & YW + V) \quad \text{w.p. } \frac{1}{2} \\ (\,V + YW, & W + YV) \quad \text{w.p. } \frac{1}{2} \end{cases}$$

where

$$W = \frac{e^{U(1+\mu)} + e^{U(1-\mu)}}{2} \tag{1}$$

$$V = \frac{e^{U(1+\mu)} - e^{U(1-\mu)}}{2}, \tag{2}$$

with $T \sim \exp(2\lambda)$ and given $T = t$, $U \sim U(0, t)$.

Conclusion

- Two-dimensional Poisson counter driven SDE
- Converges to a stationary distribution
- Understand stationary distribution as deterministic between Poisson arrivals
- Easy to do calculations - regular variation, asymptotic dependence coefficient