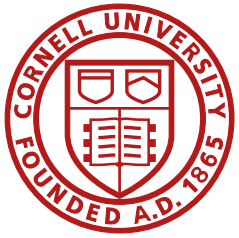


A common-neighbors-based random graph model for community structure

Emily Fischer

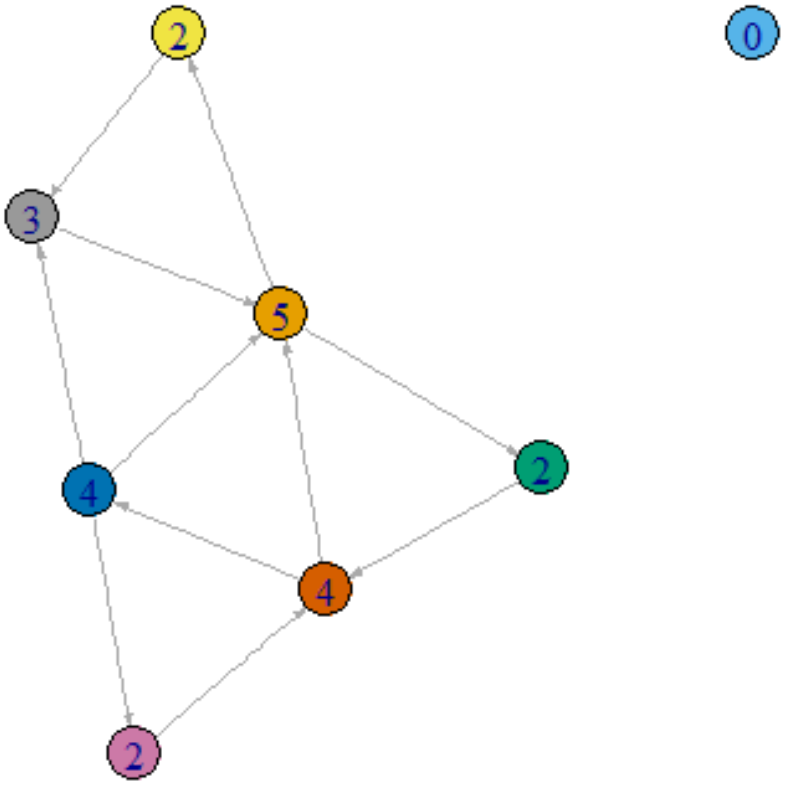
Cornell University

May 12, 2017

Outline

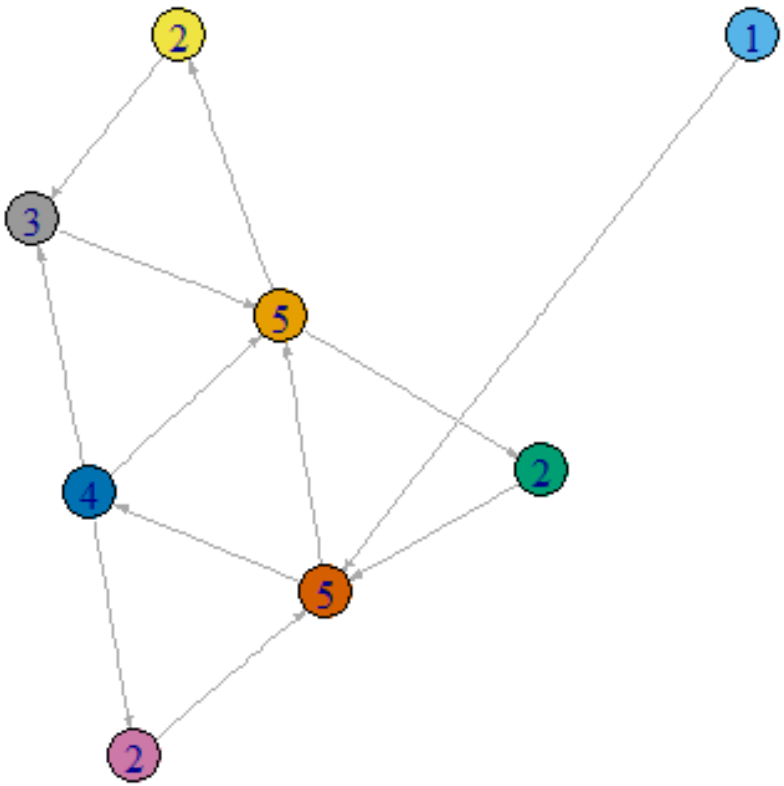
1. Introduction
   - Preferential Attachment (PA)
2. Common Neighbors Model (CN)
   - Degree distribution
   - Community structure

Preferential Attachment



- Users prefer to connect to nodes of high degree

Preferential Attachment



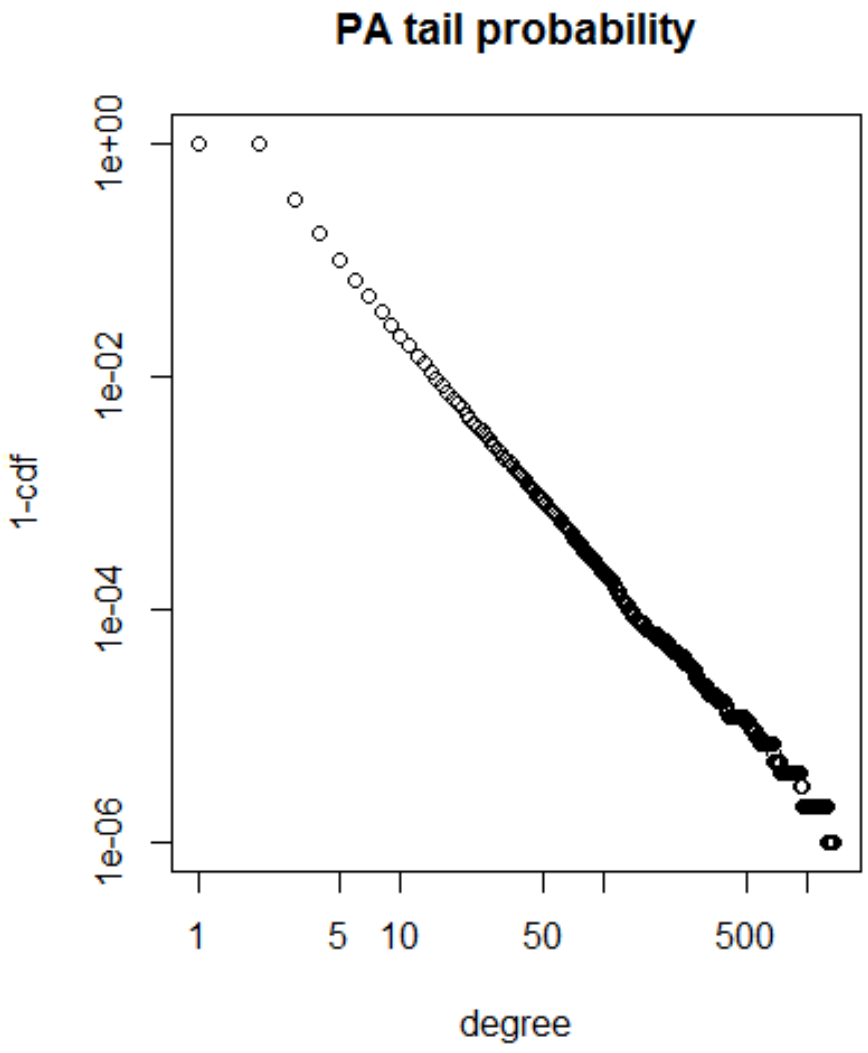
- Users prefer to connect to nodes of high degree
- Results in heavy-tailed degree distribution

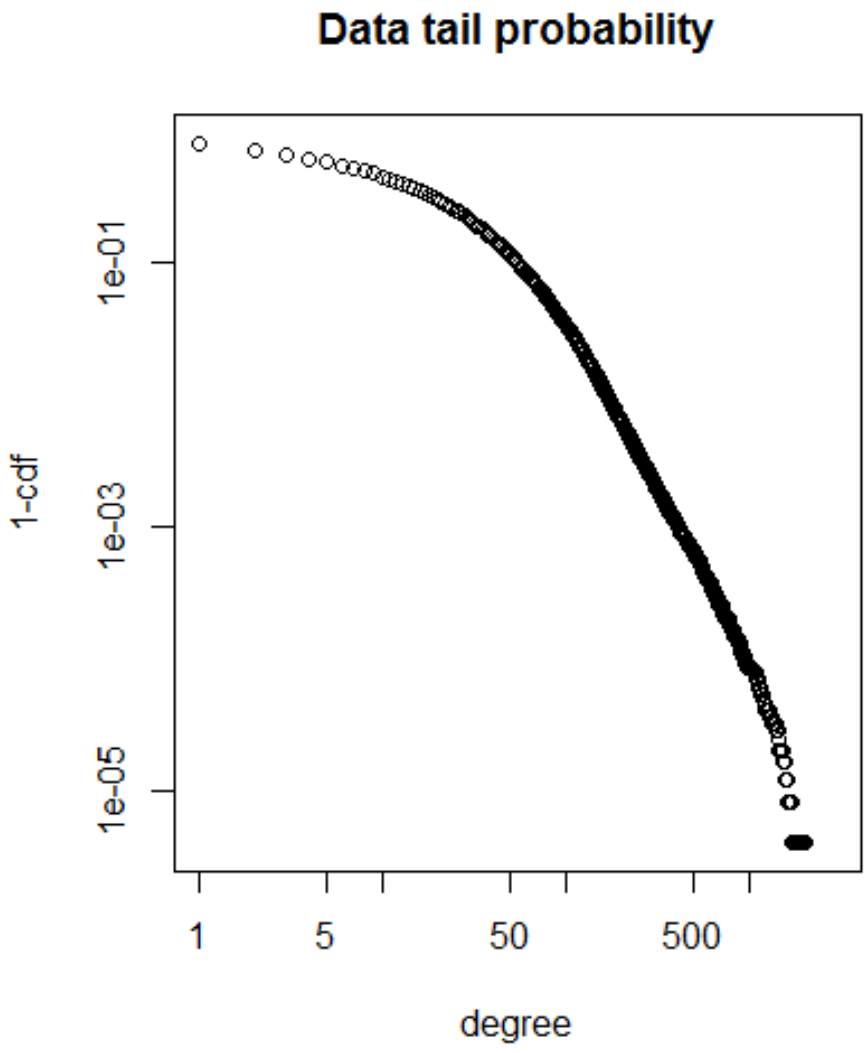
Issues with Preferential Attachmment

The LinkedIn graph

1. does NOT have a power law degree distribution
2. has "community structure"

Log-log plots of degree distribution



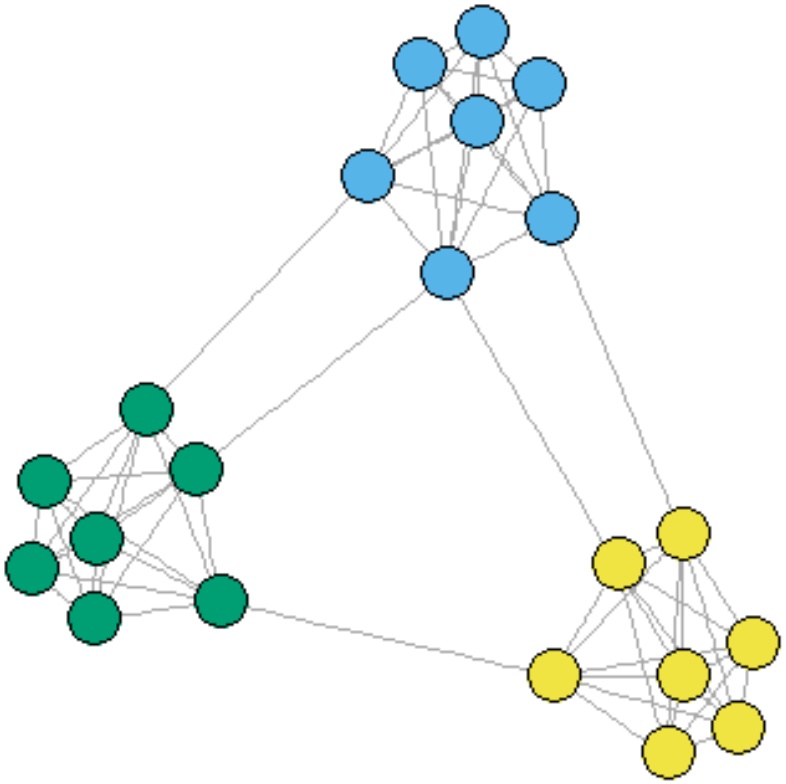


Issues with Preferential Attachmment

The LinkedIn graph

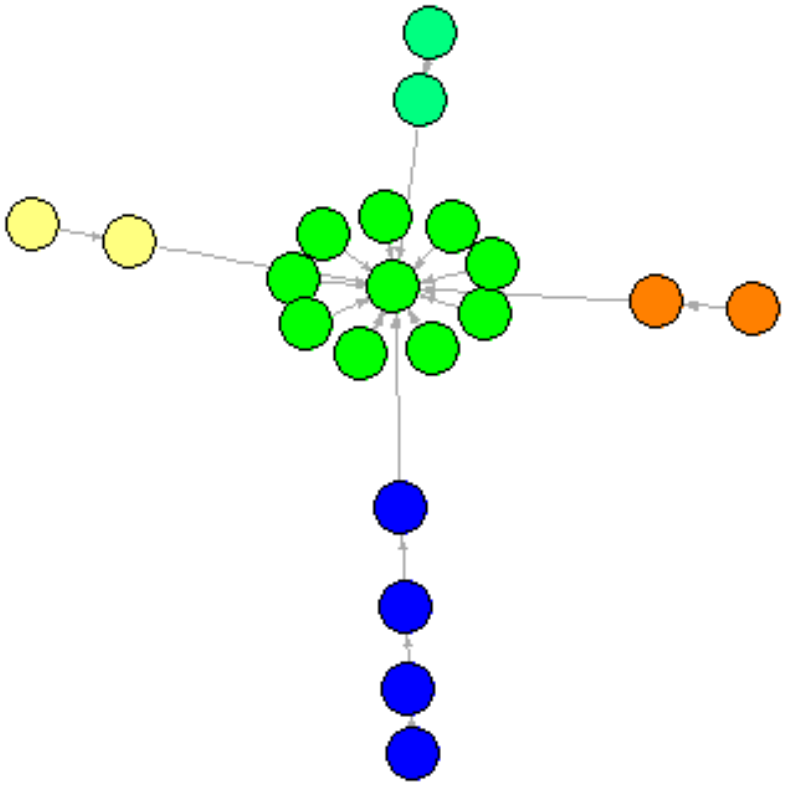
1. does NOT have a power law degree distribution
2. has "community structure"

What is "community structure"?



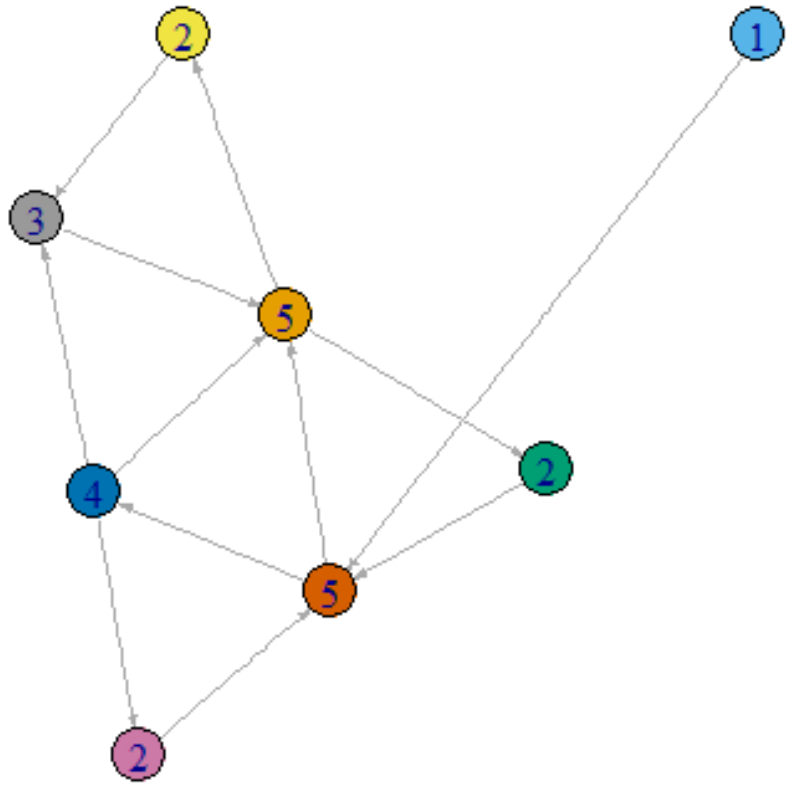
- Strong community structure
- More edges within community than between communities

What is “community structure"?

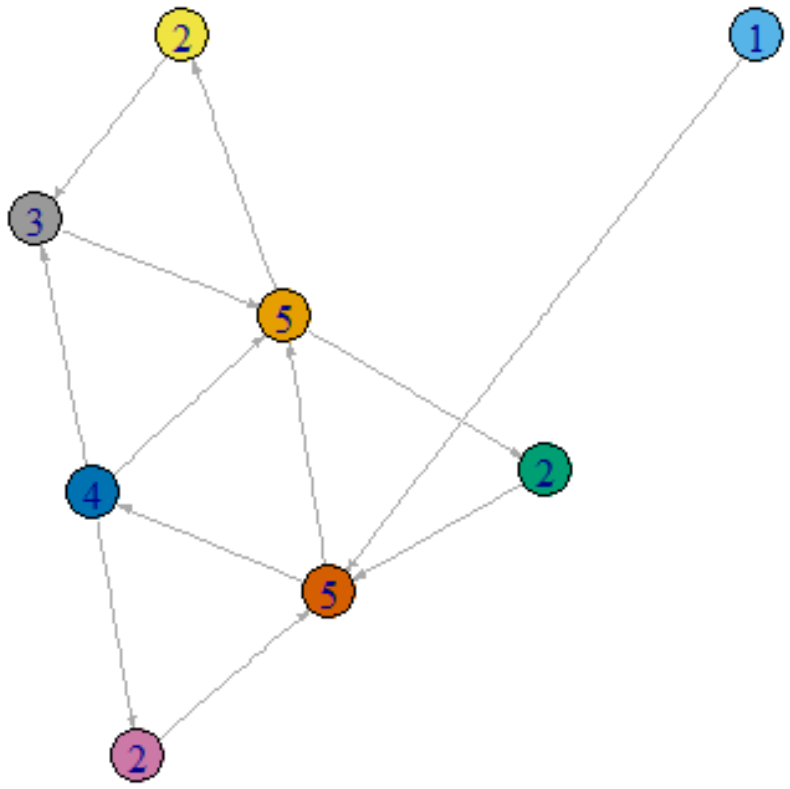


- Preferential attachment
- One central hub around high-degree node

Common Neighbors Model

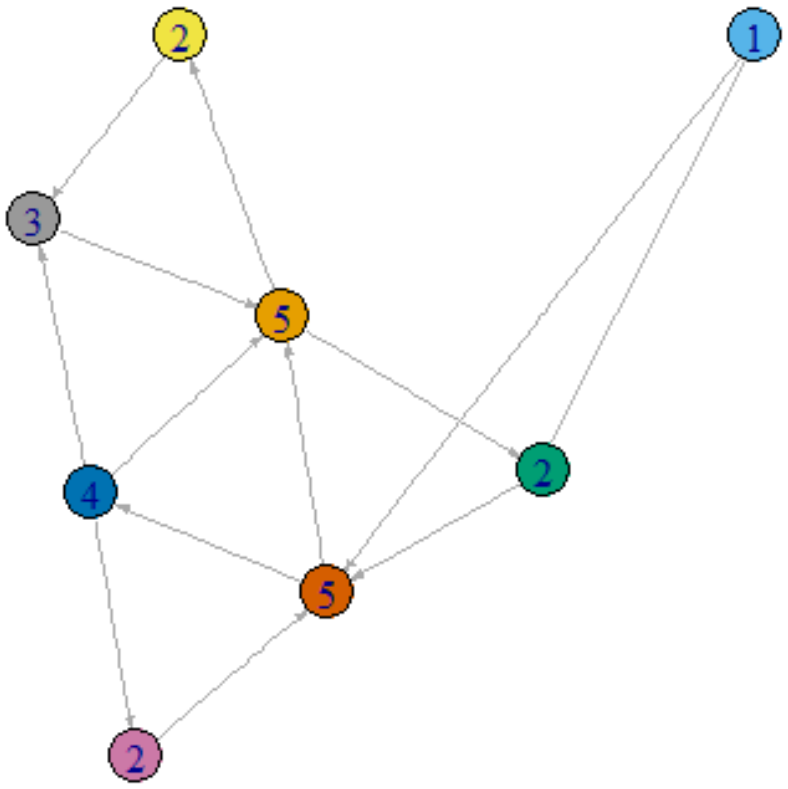


Common Neighbors Model



Users prefer to connect to nodes with whom they share many mutual friends

Common Neighbors Model



Users prefer to connect to nodes with whom they share many mutual friends

Common Neighbors Model

Sequence of graphs $(G_t)_{t \geq 0}$.

- Given graph G_t with $n(t)$ nodes and $m(t)$ edges

Common Neighbors Model

Sequence of graphs $(G_t)_{t \geq 0}$.

- Given graph G_t with $n(t)$ nodes and $m(t)$ edges
- At time $t+1$, a new node v arrives with probability α
  - If no new arrival, select v uniformly among existing nodes

Common Neighbors Model

Sequence of graphs $(G_t)_{t\geq 0}$.

- Given graph G_t with $n(t)$ nodes and $m(t)$ edges
- At time $t+1$, a new node v arrives with probability α
  - If no new arrival, select v uniformly among existing nodes
- Select receiving node w with probability proportional to number of common neighbors between v and w
  - $\Gamma_v(t)$ is the neighborhood of v at time t
  - $K_{vw}(t) = |\Gamma_v(t) \cap \Gamma_w(t)|$

$$P(\text{select } w \mid \text{sender } = v) = \frac{K_{vw}(t) + \delta}{\sum_u K_{vu}(t) + \delta n(t)}$$

Common Neighbors Model

Sequence of graphs $(G_t)_{t \geq 0}$.

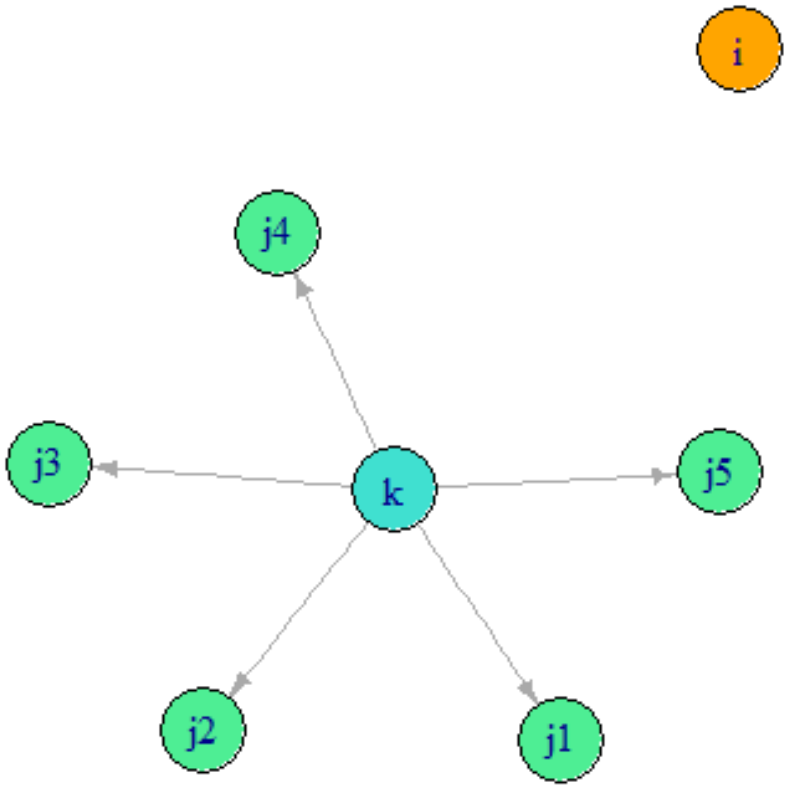
- Given graph G_t with $n(t)$ nodes and $m(t)$ edges
- At time $t+1$, a new node v arrives with probability α
  - If no new arrival, select v uniformly among existing nodes
- Select receiving node w with probability proportional to number of common neighbors between v and w
  - $\Gamma_v(t)$ is the neighborhood of v at time t
  - $K_{vw}(t) = |\Gamma_v(t) \cap \Gamma_w(t)|$

$$P(\text{select } w \mid \text{sender } = v) = \frac{K_{vw}(t) + \delta}{\sum_u K_{vu}(t) + \delta n(t)}$$

- Form directed edge (v, w).

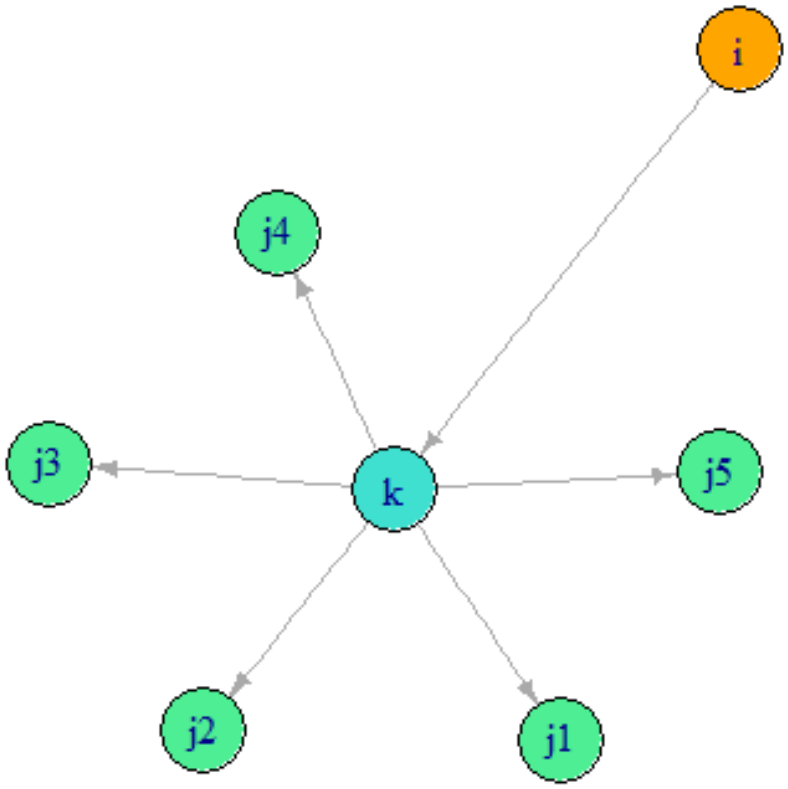
Common Neighbors Model

What does $K_{vw}(t)$ look like?



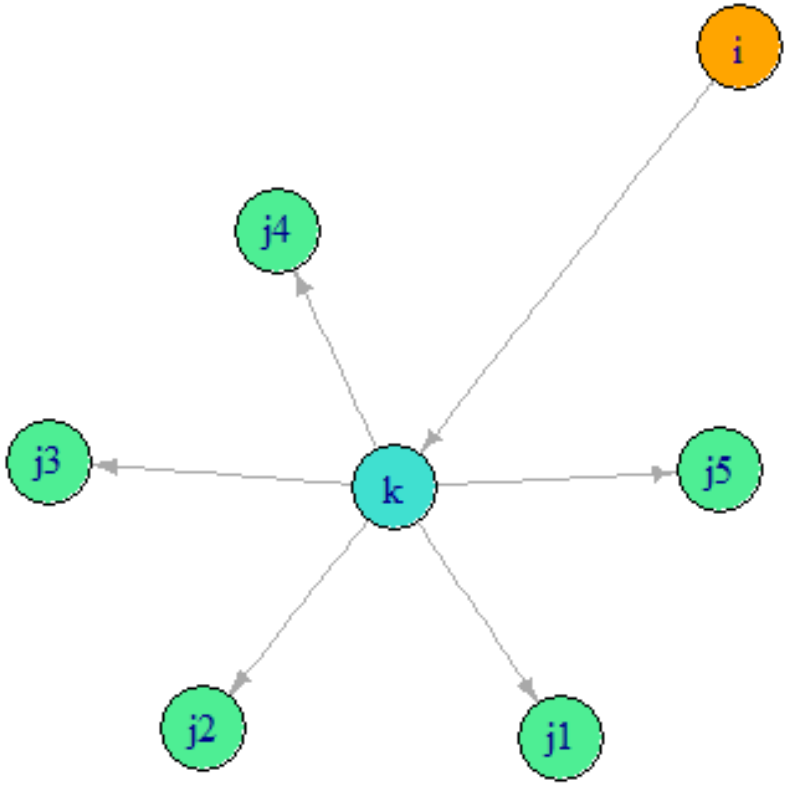
Common Neighbors Model

What does $K_{vw}(t)$ look like?



Common Neighbors Model

What does $K_{vw}(t)$ look like?



Hard to analyze - feedback

Common Neighbor Process

- Want to model evolution of $K_{ij}(t)$ on its own.
- Start at $\tilde{K}_{ij}(0) = 0$ for all pairs i, j.

Common Neighbor Process

- Want to model evolution of $K_{ij}(t)$ on its own.
- Start at $\tilde{K}_{ij}(0) = 0$ for all pairs i, j.
- Given $(\tilde{K}_{ij}(t))_{i,j \geq 0}$, at $t+1$,
  - Select i uniformly from existing nodes
  -

Common Neighbor Process

- Want to model evolution of $K_{ij}(t)$ on its own.
- Start at $\tilde{K}_{ij}(0) = 0$ for all pairs i, j.
- Given $(\tilde{K}_{ij}(t))_{i,j\geq 0}$, at $t+1$,
  - Select i uniformly from existing nodes
  - Choose $\eta = c(n(t))^{\theta}$ nodes, $j_1, j_2, \ldots, j_\eta$, preferentially with $\tilde{K}_{ij_\ell}(t)$, and increase

$$\tilde{K}_{ij_\ell}(t+1) = \tilde{K}_{ij_\ell}(t) + 1.$$

Common Neighbor Process

Let

$$N_i(t) = \sum_j \tilde{K}_{ij}(t)$$

What is the distribution of $N_i(t)$?

Common Neighbor Process

Theorem

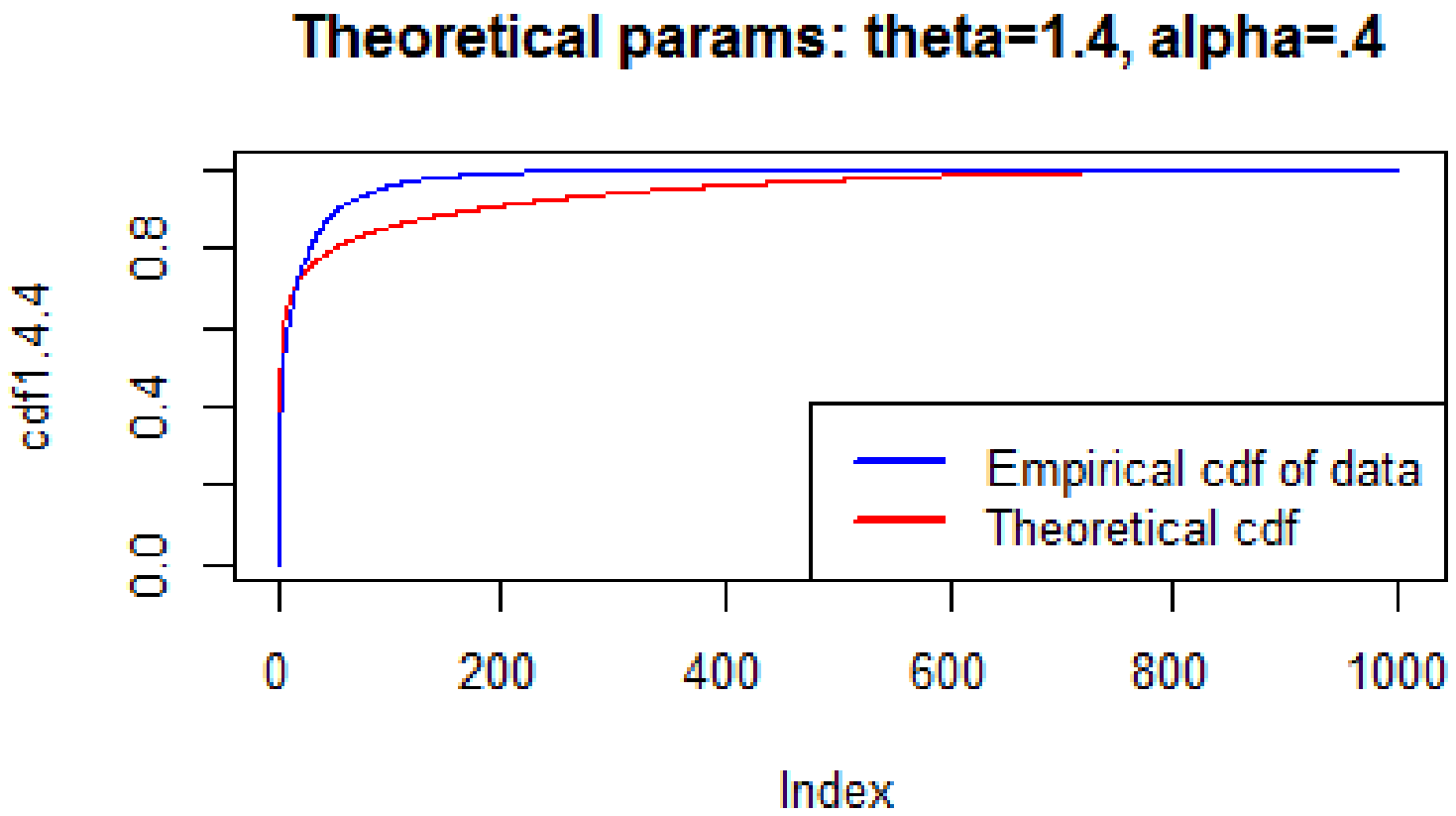
Let $N_i(t) = \sum_j \tilde{K}_{ij}(t)$. Then there exists a random variable Z_i such that

$$\frac{N_i(t)}{t^\theta} \to Z_i$$

in probability, where Z_i has characteristic function

$$\phi_Z(z) = \exp\left\{ \frac{1-\alpha}{\alpha\theta} \int_0^{\alpha^\theta} \frac{1}{t}(e^{itz} - 1)\, dt \right\}.$$

Common Neighbor Process



Common Neighbor Process

Result

- The "total common neighbors" $N_i(t)$ converges when scaled by t^θ.

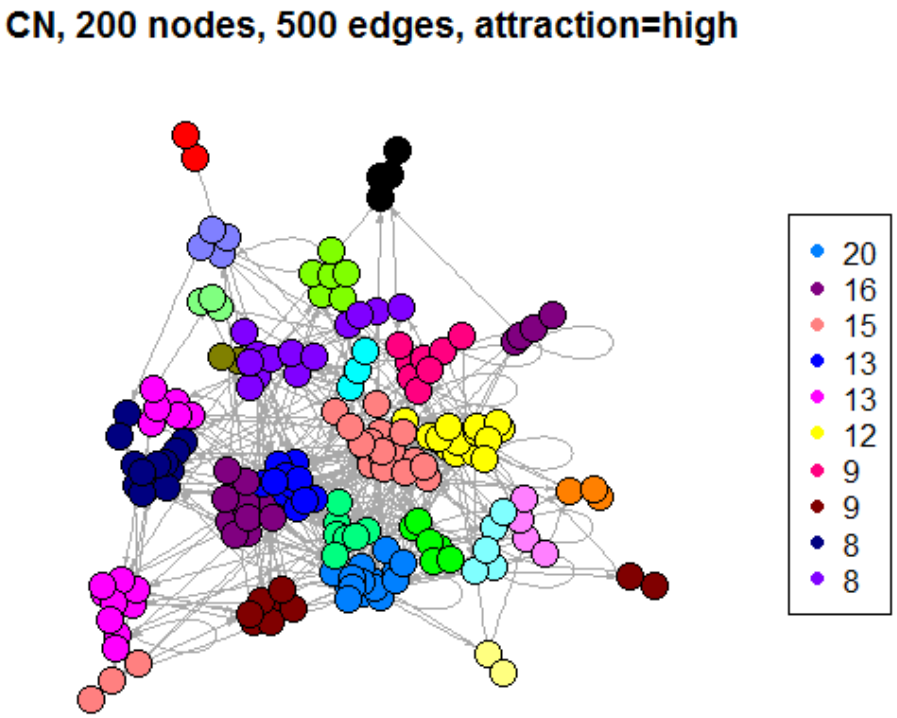
In progress/Future

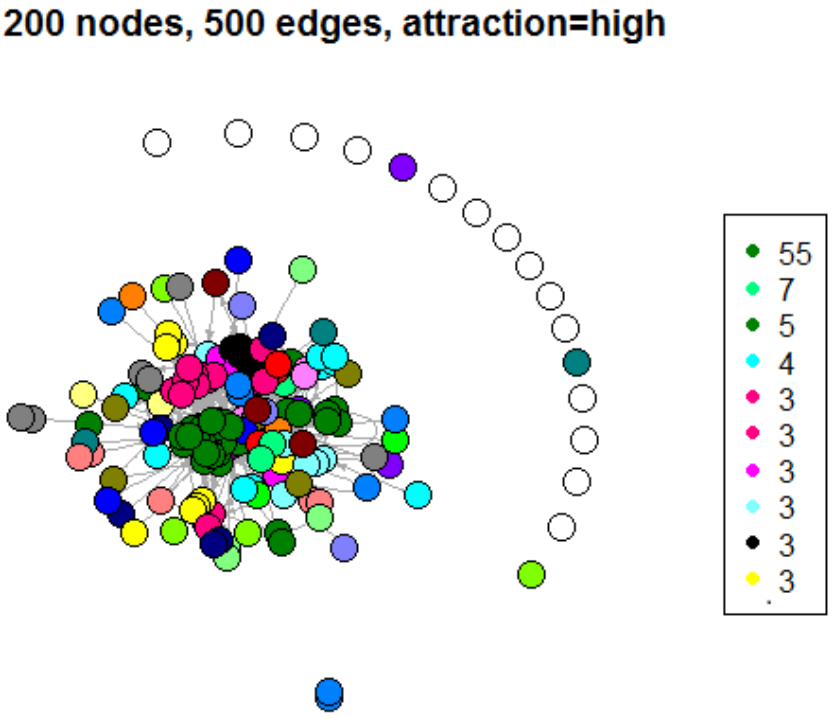
- Limiting distribution for $\tilde{K}_{ij}(t)$.
- Use these distributions to analyze degree distribution of the graph

Community Structure

- How to quantify “strong community structure”
- Compare community structure of CN and PA.

Community Structure CN vs. PA





Modularity

Definition

Given a graph partitioned into c communities, the modularity is

$$Q = \sum_{i=1}^{c} (e_{ii} - a_i^2)$$

where e_{ii} is the fraction of edges with both end vertices in community i, and a_i is the fraction of ends of edges with vertices in community i.

Community Detection

- Community detection algorithms aim to assign nodes to communities in a way that is reasonable
- Some algorithms maximize modularity: Fast-greedy (FG), Largest-eigenvector (LE)
- But there are other methods as well: Edge-betweenness (EB), Walktrap (WC).

Modularity

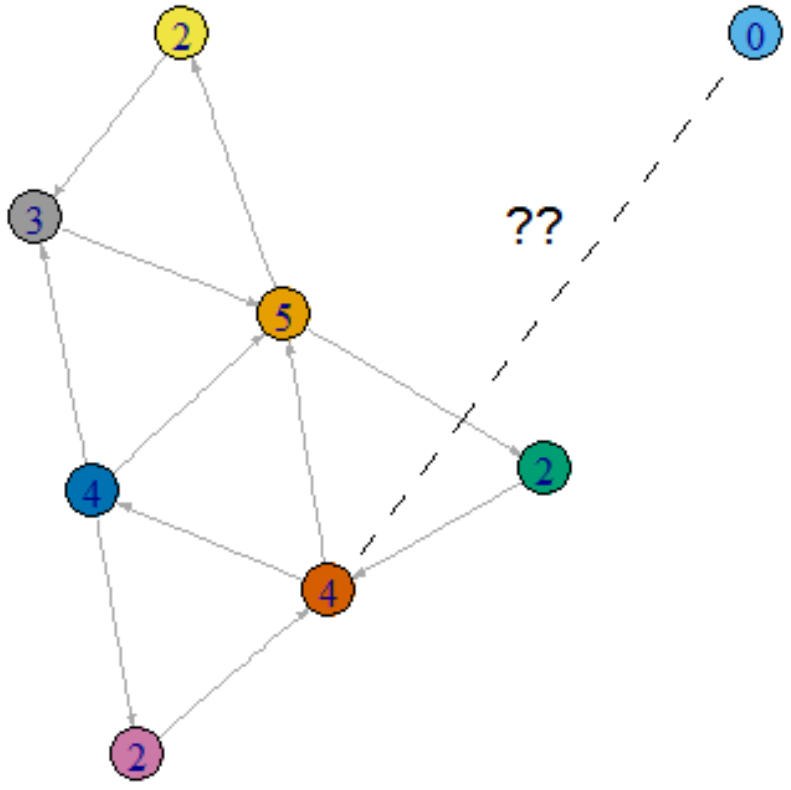
Averages of modularity over 100 trials ($\alpha = .2, \delta = .5$)

Graph	EB	FG	LE	WC
CN 500	.450	.472	.423	.401
PA 500	.276	.379	.333	.251
CN 1000	.310	.402	.350	.301
PA 1000	.103	.328	.279	.190
CN 5000	.145	.320		.176
PA 5000	.039	.277		.120

Conclusion

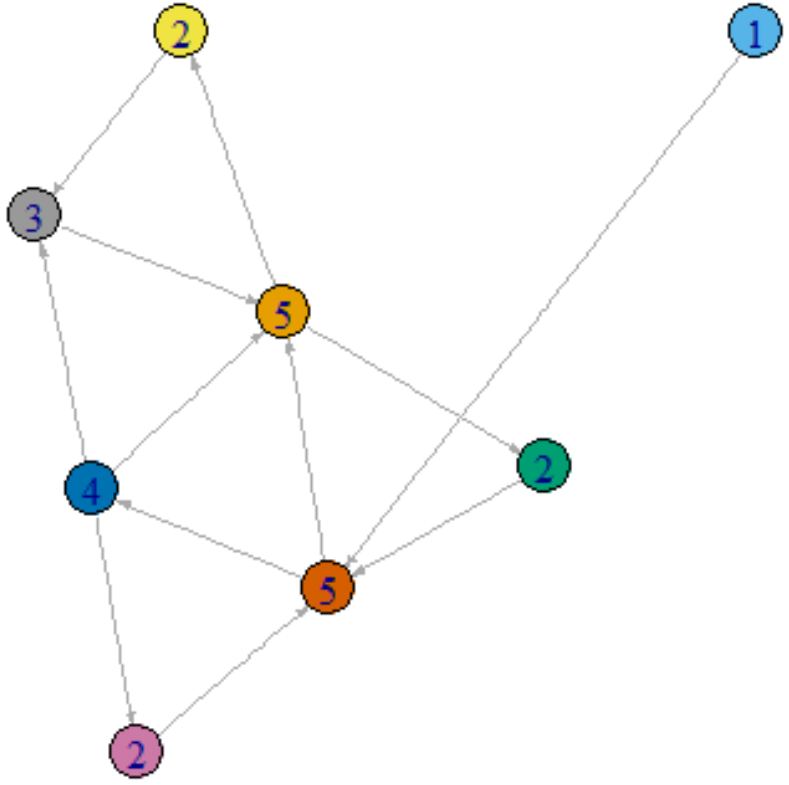
1. PA mode lacks characteristics of LinkedIn network:
   - Power-law degree distribution
   - Lack of community structure
2. Common Neighbors Model
   - Limiting distribution of $N_i(t)$ in the common neighbors process
   - Better community structure than PA

Edge Acceptance/Rejection



Node v sends an invitation to a node w.

Model 1: Edge Acceptance/Rejection



w accepts the invitation with probability $p_{vw}(t)$.

Edge Acceptance/Rejection

How can acceptance probability achieve goals of (1) non-power law degree distribution and (2) community structure?

- Rich may choose not to get richer
- Probability of acceptance based on communities

Edge Acceptance/Rejection

How can acceptance probability achieve goals of (1) non-power law degree distribution and (2) community structure?

- Rich may choose not to get richer: $p_{vw}(t) \downarrow 0$
- Probability of acceptance based on communities

Edge Acceptance/Rejection

How can acceptance probability achieve goals of (1) non-power law degree distribution and (2) community structure?

- Rich may choose not to get richer: $p_{vw}(t) \downarrow 0$
- Probability of acceptance based on communities:

$$p_{vw}(t) = \begin{cases} p & C_v = C_w \\ q & C_v \neq C_w. \end{cases}$$

Edge Acceptance/Rejection

For now, constant acceptance probability

$$p_{vw}(t) = p \quad \text{for all } v, w \text{ and } t \geq 0.$$