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# ASYMPTOTIC THEORY FOR THE SAMPLE COVARIANCE MATRIX OF A HEAVY-TAILED MULTIVARIATE TIME SERIES

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Abstract. In this paper we give an asymptotic theory for the eigenvalues of the sample covariance matrix of a multivariate time series. The time series constitutes a linear process across time and between components. The input noise of the linear process has regularly varying tails with index $\alpha \in (0,4)$; in particular, the time series has infinite fourth moment. We derive the limiting behavior for the largest eigenvalues of the sample covariance matrix and show point process convergence of the normalized eigenvalues. The limiting process has an explicit form involving points of a Poisson process and eigenvalues of a non-negative definite matrix. Based on this convergence we derive limit theory for a host of other continuous functionals of the eigenvalues, including the joint convergence of the largest eigenvalues, the joint convergence of the largest eigenvalue and the trace of the sample covariance matrix, and the ratio of the largest eigenvalue to their sum.

## 1. Introduction

In the setting of classical multivariate statistics or multivariate time series, the data consist of $n$ observations of $p$-dimensional random vectors, where $p$ is relatively small compared to the sample size $n$. With the recent advent of large data sets, the dimension $p$ can be large relative to the sample size and hence standard asymptotics, assuming $p$ is fixed relative to $n$ may provide misleading results. Structure in multivariate data is often summarized by the sample covariance matrix. For example, principal component analysis, extracts principal component vectors corresponding to the largest eigenvalues. Consequently, there is a need to study asymptotics of the largest eigenvalues of the sample covariance matrix. In the case of $p$ fixed and the $p \times n$ data matrix consists of iid N(0,1) observations, Anderson [1] showed that the largest eigenvalue is asymptotically normal. In a now seminal paper, Johnstone [11] showed that if $p_n \to \infty$ at the rate $p_n/n \to \gamma \in (0,\infty)$, then the largest eigenvalues, suitable normalized, converges to the *Tracy-Widom* distribution with $\beta = 1$. Johnstone's result has been generalized by Tao and Vu [19] where only 4 moments are needed to determine the limit. The theory for the largest eigenvalues of sample covariance and Wigner matrices based on heavy tails is not as well developed as in the light tailed case. The largest eigenvalues of sample covariance matrices with iid entries that are regularly varying with index $-\alpha$ were studied by Soshnikov [18] for the $\alpha \in (0,2)$ case and subsequently extended in Auffinger et el. [2] to the $\alpha \in (2,4)$ case. They showed that the point process of eigenvalues, normalized by the square of the $1-(np)^{-1}$ quantile converges in distribution to a Poisson point process with intensity $(\alpha/2)x^{-\alpha/2-1}$, provided $p/n \to \gamma$, where $\gamma \in (0,1)$. These results were extended in Davis et al. [7] to the case where the rows of the data matrix are iid linear heavy-tailed processes. They also had more general growth conditions on $p_n$ in the case of iid entries and $\alpha \in (0,2)$.

In this paper, we study the asymptotic behavior of the largest eigenvalues of the sample covariance matrices of a multivariate time series. The time series is assumed to be heavy-tailed and linearly dependent in time and between the components. Even though [7] allowed for some

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dependence between the rows, it was somewhat contrived in that the rows were assumed to be conditionally independent given a random process. To our knowledge, the present paper is the first to consider bona fide dependence among the components in the time series which renders a multivariate analysis, such as PCA, meaningful. Allowing for dependence between the rows can appreciably impact the limiting behavior of the largest eigenvalues. Instead of obtaining a Poisson point process as the limit of the extreme eigenvalues, we now get a "cluster" Poisson point process. That is, the limit can be described by a Poisson point process in which each point produces a "cluster" of points. The clusters are determined via the eigenvalues of an auxiliary *covariance matrix* that is constructed from the linear filter weights. Interestingly, the limit point process is identical to the limit point process derived by Davis and Resnick [6] for the extremes of a linear process. One of the striking differences in the limit theory between the independent and dependent row cases is the limiting behavior of the ratio of the second largest eigenvalue, $\lambda_{(2)}$ to the largest eigenvalue $\lambda_{(1)}$ of the sample covariance matrix. In the independent row case,

$$\lambda_{(2)}/\lambda_{(1)} \xrightarrow{d} U^{\alpha/2}\,,$$

where $U$ is a uniform random variable on $(0,1)$ and $\alpha \in (0,2)$ is the index of regular variation. Now if the rows are dependent, then the limit random variable corresponds to a truncated uniform, i.e., there exists a constant $c \in [0,1)$ such that the limit has the form $c^{\alpha/2} I_{\{U<c\}} + U^{\alpha/2} I_{\{U\geq c\}}$. The constant $c$ is determined from the eigenvalues of the auxiliary covariance matrix.

To make the model precise, consider a field of iid random variables $(Z_{it})_{i,t\in\mathbb{Z}}$, a double array of real numbers $(h_{kl})_{k,l\in\mathbb{Z}}$ such that $h_{kl}=0$ if $k$ or $l$ are negative and construct an infinite-dimensional time series,

$$X_{it} = \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} h_{kl} Z_{i-k,t-l}\,, \quad i,t\in\mathbb{Z}\,. \tag{1.1}$$

We also assume that a generic element $Z$ of the $Z$-field satisfies the regular variation and tail balance condition

$$P(Z>x) \sim p_+ \frac{L(x)}{x^\alpha} \quad \text{and} \quad P(Z\leq -x) \sim p_- \frac{L(x)}{x^\alpha}\,, \quad x\to\infty\,, \tag{1.2}$$

for some tail index $\alpha>0$, where $p_+,p_-\geq 0$ with $p_+ + p_- = 1$ and $L$ is a slowly varying function. To ensure the a.s. absolute convergence of the series (1.1) we will need further conditions on $(h_{kl})$ to be discussed later.

Consider the $p\times n$ data matrix

$$\mathbf{X}_n = (X_{it})_{i=1,\ldots,p,t=1,\ldots,n}\,, \quad n\geq 1\,, \tag{1.3}$$

where $p=p_n$ is an integer sequence such that $p_n\to\infty$.

The main focus of study in this paper is the asymptotic behavior of the eigenvalues $n$ times the sample covariance matrix $\mathbf{X}_n\mathbf{X}_n'$ in the case $\alpha\in(0,2)$ and its centered version $\mathbf{X}_n\mathbf{X}_n' - E\mathbf{X}_n\mathbf{X}_n'$ in the case $\alpha\in(2,4)$. Our main result, Theorem 3.1, yields an approximation for the largest eigenvalues of the sample covariance matrices, showing that these eigenvalues are to a large extent determined by the order statistics of $D_1,\ldots,D_p$, where, for $n\geq 1$, we define the iid sequence

$$D_s = D_s^{(n)} = \sum_{t=1}^{n} Z_{st}^2\,, \quad s\in\mathbb{Z}\,. \tag{1.4}$$

A consequence of this approximation is the point process convergence of the normalized eigenvalues of the sample covariance matrices. Based on the point process convergence, the continuous mapping theorem yields a variety of asymptotic results for the largest eigenvalues of the sample covariance matrix as well as joint limit theory for the trace and the largest eigenvalue. In particular, we show that the ratio of the largest eigenvalue to their sum converges in distribution to the ratio of a max-stable to a sum-stable random variable. In the special case when the filter $(h_{kl})$ is separable, $h_{kl} = \theta_k c_l$, the limit ratio does not depend on the filter weights $(\theta_k)$, $(c_l)$. As a further special case, if the time series consists of iid vectors with linear dependence between the components the limit behavior of the eigenvalues is the same as that for iid components.

The paper is organized as follows. In Section 2 we introduce the notation and various conditions used throughout the paper. In Section 3 we formulate the approximation results in terms of $(D_s)$ for the eigenvalues of the sample covariance matrices and discuss the conditions. In Section 4 we derive the convergence of the point processes of the normalized eigenvalues to a cluster Poisson process and prove results for various functionals of the eigenvalues. We also give examples illustrating the results. The proof of the main result, Theorem 3.1, is given in Section 5. The Appendix contains various useful results about large deviations for sums of iid random variables as well as point process convergence results for iid sequences of sums of iid heavy-tailed random variables. These results are needed in the proof of Theorem 3.1 and its corollaries.

## 2. Preliminaries

In this section, we introduce some new notation and conditions to be used throughout the paper.

### 2.1. **Notation.**

*Eigenvalues of the sample covariance matrix.* Fix $n \geq 1$. We denote the eigenvalues of $\mathbf{X}_n\mathbf{X}_n'$ in the case $\alpha \in (0,2)$ by $\lambda_1, \dots, \lambda_p$, and we use the same notation in the case $\alpha \in (2,4)$ for the eigenvalues of $\mathbf{X}_n\mathbf{X}_n' - E\mathbf{X}_n\mathbf{X}_n'$. In this notation, we suppress the dependence of the eigenvalues on $n$.

*The matrix $M$.* In order to describe the limit behavior of the eigenvalues, we need to introduce the eigenvalues of a matrix $M$ determined by the coefficients $(h_{kl})_{k,l\geq 0}$.

Set $\mathbf{h}_i = (h_{i0}, h_{i1}, \dots)'$ and define the $\infty \times \infty$ matrix $H = (\mathbf{h}_0, \mathbf{h}_1, \dots)$ and put

$$M = HH'\,. \tag{2.1}$$

In particular, the $(i,j)$-th entry of $M$ is

$$M_{ij} = \mathbf{h}_i\mathbf{h}_j' = \sum_{l=0}^{\infty} h_{il}h_{jl}\,, \quad i,j = 0,1,\dots\,.$$

By construction, $M$ is symmetric and non-negative definite, hence it has non-negative eigenvalues denoted by

$$v_1 \geq v_2 \geq \cdots\,. \tag{2.2}$$

Let $r$ be the rank of $M$ so that $v_r > 0$ while $v_{r+1} = 0$ if $r$ is finite, otherwise $v_i > 0$ for all $i$. For later reference, note that under condition (2.7) on $(h_{kl})$

$$\operatorname{tr}(M) = \sum_{i=0}^{\infty} M_{ii} = \sum_{i=1}^{\infty} v_i = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} h_{kl}^2 < \infty\,. \tag{2.3}$$

Therefore all eigenvalues $v_i$ are finite and the ordering (2.2) is justified.

*Order statistics of the iid sequence $(D_s)$ in* (1.4). For given $n \geq 1$, denote the order statistics of $D_1, \dots, D_p$ by

$$D_{(p)} = D_{L_p} \leq \cdots \leq D_{(1)} = D_{L_1}\,, \tag{2.4}$$

where we assume that $(L_1, \dots, L_p)$ is a permutation of $(1, \dots, p)$.

By $D$, we denote a generic element of $(D_i)$. Assuming that $ED$ is finite, write $\widetilde{D}_s = |D_s - ED|$, $s \in \mathbb{Z}$, and consider the corresponding order statistics

$$\widetilde{D}_{(p)} = \widetilde{D}_{\ell_p} \leq \cdots \leq \widetilde{D}_{(1)} = \widetilde{D}_{\ell_1}\,, \tag{2.5}$$

where we again assume that $(\ell_1,\ldots,\ell_p)$ is a permutation of $(1,\ldots,p)$.

*Normalizing sequence* $(a_{np}^2)$. We will normalize the eigenvalues of the matrices $(\mathbf{X}_n\mathbf{X}_n')$ by the sequence $(a_{np}^2)$ which is derived from the regularly varying tails of the iid noise $(Z_{it})$ with generic element $Z$. We define $(a_k)$ by

$$P(|Z|>a_k)\sim k^{-1}\,,\quad k\to\infty\,. \tag{2.6}$$

2.2. **Conditions on** $(h_{kl})$**.** To ensure the a.s. absolute convergence of the $X_{it}$'s defined in (1.1) with noise satisfying (1.2) we assume

$$\sum_{l=0}^\infty\sum_{k=0}^\infty |h_{kl}|^\delta<\infty \tag{2.7}$$

for some $\delta\in(0,\min(\alpha/2,1))$ and the considered $\alpha\in(0,4)$. In addition, we will need the following technical assumption for the same $\alpha$-values and $\varepsilon>0$ arbitrarily close to zero:

$$\sum_{t=0}^\infty\Big(\sum_{l=t}^\infty|h_{kl}|\Big)^{\alpha/2-\varepsilon}<\infty\,,\quad k=0,1,\ldots,. \tag{2.8}$$

The latter condition is satisfied if $\alpha\in(0,4)$ and

$$\sum_{l=0}^\infty l^{2/\alpha+\varepsilon'}\,|h_{kl}|<\infty\,,\quad k=0,1,\ldots,, \tag{2.9}$$

for $\varepsilon'>0$ arbitrarily close to zero. Indeed, write $\beta=\alpha/2-\varepsilon\in(0,2)$, $c_t=\sum_{l=t}^\infty|h_{kl}|$ for fixed $k$, suppressing the dependence on $k$ in the notation, and let $(Y_t)$ be an iid sequence of symmetric $\beta$-stable random variables. Then for any small $\varepsilon''>0$ such that $1/\beta+\varepsilon''=2/\alpha+\varepsilon'$,

$$\begin{aligned}Y_0\Big(\sum_{t=0}^\infty c_t^\beta\Big)^{1/\beta} &\overset{d}{=} \sum_{t=0}^\infty c_tY_t\\ &= \sum_{l=0}^\infty|h_{kl}|\sum_{t=1}^l Y_t\\ &= \sum_{l=0}^\infty|h_{kl}|l^{1/\beta+\varepsilon''}\Big[l^{-1/\beta-\varepsilon''}\sum_{t=1}^l Y_t\Big]\,.\end{aligned}$$

But $l^{-1/\beta-\varepsilon''}\sum_{t=1}^l Y_t\overset{\text{a.s.}}{\to}0$; see Petrov [14], Theorem 6.9. Therefore condition (2.9) implies that $\sum_{t=0}^\infty c_t^\beta<\infty$, i.e. (2.8) holds.

2.3. **Growth conditions on** $(p_n)$**.** Recall that $p=p_n\to\infty$ is the number of rows in the matrix $\mathbf{X}_n$. We need conditions on the growth of $(p_n)$ to ensure the convergence of the normalized eigenvalues of the sample covariance matrix.

Let $(Z_t)$ be iid copies of $Z$.

*Condition* $\mathbf{C}_\alpha$*.*

- For $\alpha\in(0,1)$, assume

$$\lim_{n\to\infty}p\,[n\,p\,P(|Z_1Z_2|>a_{np}^2)]=0\,, \tag{2.10}$$

- For $\alpha=1$ and $E|Z|=\infty$, assume (2.10) and

$$\lim_{n\to\infty}p\,[n\,p\,a_{np}^{-2}E|Z_1Z_2|I_{\{|Z_1Z_2|\le a_{np}^2\}}]=0\,, \tag{2.11}$$

- For $\alpha \in (1,2)$ or $\alpha = 1$ and $E|Z| < \infty$, assume there exists a $\gamma \in (\alpha, 2)$ arbitrarily close to $\alpha$ such that

$$\lim_{n\to\infty} p^{\gamma}\,[n\,p\,P(|Z_1Z_2| > a_{np}^2)] = 0\,, \tag{2.12}$$

- For $\alpha \in (2,4)$, assume there exists a $\gamma \in (\alpha, 4)$ arbitrarily close to $\alpha$ such that

$$\lim_{n\to\infty} n^{\gamma/2-1} p^{\gamma}[n\,p\,P(|Z_1Z_2| > a_{np}^2)] = 0\,. \tag{2.13}$$

Note that the conditions (2.10)–(2.13) that restrict the growth of the integer sequence $(p_n)$ depend on the slowly varying functions $L$ and $L_2$ in the tails $P(|Z| > x) \sim L(x)x^{-\alpha}$ and $P(|Z_1Z_2| > x) = x^{-\alpha}L_2(x)$; see Embrechts and Goldie [9] for a proof of the second identity. We also know that $a_n = n^{1/\alpha}\ell(n)$ for a suitable slowly varying function $\ell$. With this information in mind, we may now verify the various conditions for certain choices of $(p_n)$.

*The case $\alpha \in (0,1)$.* For arbitrarily small positive $\epsilon$ and large $n$, we have from (2.10) (see Bingham et al. [4])

$$p\,[n\,p\,P(|Z_1Z_2| > a_{np}^2)] \le cnp^2(np)^{-2+\epsilon} = cn^{-1+\epsilon}p^{\epsilon}\,. \tag{2.14}$$

The right-hand side converges to zero for sufficiently small $\epsilon$ if $p$ has polynomial growth, i.e., $p = O(n^{\beta})$, $\beta > 0$, in particular, one can choose $p/n \to c \in (0,\infty)$ for some constant $c$. *Here and in what follows, we write $c$ for any constant whose value is not of interest.*

If more information is available the rate of $p$ in (2.10) can be much faster. For example, if $Z$ has a Pareto distribution then $\ell$ and $L$ are constants, $L_2(x) \sim c\,\log x$ as $x \to \infty$, $a_{np} \sim c(np)^{1/\alpha}$ as $n \to \infty$ and we have

$$p\,[n\,p\,P(|Z_1Z_2| > a_{np}^2)] \sim c\,n^{-1}\log(np)\,.$$

In this case, one can choose $p = O(\mathrm{e}^{\,c_n})$ for any $(c_n)$ such that $n^{-1}c_n \to 0$ as $n \to \infty$.

*The case $\alpha = 1$ and $E|Z| = \infty$.* Condition (2.10) can be verified as in the case $\alpha \in (0,1)$. In addition, one has to check (2.11). Note that $x^{-1}E|Z_1Z_2|I_{\{|Z_1Z_2|\le x\}}/P(|Z_1Z_2| > x)$ is a slowly varying function converging to infinity as $x \to \infty$ (Bingham et al. [4], (1.5.8)) and therefore for arbitrarily small $\epsilon > 0$,

$$n\,p^2\,a_{np}^{-2}E|Z_1Z_2|I_{\{|Z_1Z_2|\le a_{np}^2\}} \le c\,np^2(np)^{-2+\epsilon} = c(np)^{-1+\epsilon}p\,.$$

The right-hand side coincides with the bound in (2.14). Thus (2.11) holds under conditions on $p$ similar to those for (2.10).

The Pareto distribution with tail $P(Z > x) = x^{-1}$, $x > 1$, belongs to the considered class of distributions. In this case,

$$n\,p^2\,a_{np}^{-2}E|Z_1Z_2|I_{\{|Z_1Z_2|\le a_{np}^2\}} \sim cn^{-1}\log^2(np)\,,$$

and (2.11) holds for $p = O(\mathrm{e}^{\,c_n})$ and any $(c_n)$ such that $n^{-1}c_n^2 \to 0$ as $n \to \infty$.

*The cases $\alpha \in (1,2)$ and $\alpha = 1$, $EZ = 0$.* Condition (2.12) holds if

$$p^{\gamma}[n\,p\,P(|Z_1Z_2| > a_{np}^2)] = (np)^{-1}L_2(a_{np}^2)p^{\gamma} \le (np)^{-1+\epsilon}p^{\gamma} \to 0\,,\quad n\to\infty\,,$$

for any $\epsilon > 0$ arbitrarily close to zero and $\gamma > \alpha$ arbitrarily close to $\alpha$. These conditions are satisfied if $p/n \to c \in (0,\infty)$ for some constant $c$.

For $Z$ with a Pareto-like tail $P(Z > x) \sim cx^{-\alpha}$, $\alpha \in (1,2)$, (2.12) boils down to verifying $n^{-1}p^{\gamma-1}\log(np) \to 0$ as $n \to \infty$ and we can choose $p = n^c$ for positive $c < (\gamma-1)^{-1}$.

*The case $\alpha \in (2,4)$.* Condition (2.13) is satisfied if

$$L_2(a_{np}^2)n^{\gamma/2-2}p^{\gamma-1} \le (np)^{\epsilon}p^{\gamma-1}n^{\gamma/2-2} \to 0\,,\quad n\to\infty\,,$$

for any $\epsilon > 0$ arbitrarily close to zero and $\gamma > \alpha$ arbitrarily close to $\alpha$. The latter condition is satisfied for integer sequences $p = O(n^\delta)$ for $\delta \in (0, (4-\alpha)/[2(\alpha-1)])$, excluding sequences with $p/n \to c$ for some positive constant $c$.

## 3. The main result

Now we are ready to formulate the main result of this paper.

**Theorem 3.1.** *Consider the random matrices* $(\mathbf{X}_n)$ *defined in* (1.1) *and* (1.3). *Assume the following conditions.*

- *The regular variation and tail balance condition* (1.2) *with index* $\alpha \in (0,4)$ *on the distribution of* $Z$, *and* $EZ = 0$ *if* $E|Z| < \infty$.
- *The summability conditions* (2.7) *and* (2.8) *on the coefficients* $(h_{kl})$.
- *The normalizing sequence* $(a_n)$ *satisfies* (2.6).
- *The number of the rows* $p_n$ *of* $\mathbf{X}_n$ *satisfies the growth condition* $\mathbf{C}_\alpha$.
- *Let* $k = k_p \to \infty$ *be any integer sequence such that* $k^2 = o(p)$ *as* $n \to \infty$.

*Then the following statements hold.*

(1) *If* $\alpha \in (0,2)$, *then*

$$a_{np}^{-2} \max_{i=1,\dots,p} \left|\lambda_{(i)} - \delta_{(i)}\right| \xrightarrow{P} 0\,, \quad n \to \infty\,, \tag{3.1}$$

*where* $\lambda_{(1)} \geq \cdots \geq \lambda_{(p)}$ *a.s. are the order statistics of the eigenvalues* $\lambda_1, \dots, \lambda_p$ *of* $\mathbf{X}_n\mathbf{X}_n'$, $\delta_{(1)} \geq \cdots \geq \delta_{(p)}$ *are the ordered values from the set* $\{D_{(i)}v_j, i = 1,\dots,k, j = 1,2,\dots\}$; *cf.* (2.2) *and* (2.4).

(2) *If* $\alpha \in (2,4)$, *then*

$$a_{np}^{-2} \max_{i=1,\dots,p} \left|\widetilde{\lambda}_{(i)} - \widetilde{\delta}_{(i)}\right| \xrightarrow{P} 0\,, \quad n \to \infty\,, \tag{3.2}$$

*where* $\widetilde{\lambda}_{(1)} \geq \cdots \geq \widetilde{\lambda}_{(p)}$ *are the ordered eigenvalues* $\lambda_1,\dots,\lambda_p$ *of* $\mathbf{X}_n\mathbf{X}_n' - E\mathbf{X}_n\mathbf{X}_n'$ *and* $\widetilde{\delta}_{(1)} \geq \cdots \geq \widetilde{\delta}_{(p)}$ *are the ordered components of* $\{(D_{\ell_i} - ED)v_j, i = 1,\dots,k, j = 1,2,\dots\}$; *cf.* (2.2) *and* (2.5).

**Remark 3.2.** Discussions of the conditions on $(h_{kl})$ and the growth of $(p_n)$ are given in Sections 2.2 and 2.3, respectively. The case $\alpha = 2$ can be treated by similar methods but then centering with the expected values of the entries $\mathbf{X}_n\mathbf{X}_n'$ truncated at a suitable level, such as $a_{np}^2$, would become necessary. We decided to exclude this special case since it requires additional technical arguments that are geared directly to $\alpha = 2$ and do not provide further insight to the other cases.

**Remark 3.3.** If $\alpha \in (2,4)$ the matrix $\mathbf{X}_n\mathbf{X}_n' - E\mathbf{X}_n\mathbf{X}_n'$ may not be non-negative definite. Therefore its eigenvalues $(\lambda_i)$ and the corresponding sequence $(\widetilde{\lambda}_{(i)})$ ordered according to their absolute values are not necessarily non-negative. However, the theory in Section 4 ensures that the point process of the eigenvalues of the normalized and centered sample covariance matrices $a_{np}^{-2}(\mathbf{X}_n\mathbf{X}_n' - E\mathbf{X}_n\mathbf{X}_n')$ converges weakly to a Poisson process with support on $(0,\infty)$.

**Remark 3.4.** An immediate consequence of Theorem 3.1 is an approximation for the largest (in absolute value) eigenvalues of the centered sample covariance matrix $n^{-1}(\mathbf{X}_n\mathbf{X}_n - E\mathbf{X}_n\mathbf{X}_n)$ in the case $\alpha \in (2,4)$. If $n^{-1}a_{np}^2 \to 0$ then

$$n^{-1} \sup_{i \leq k} \left|\widetilde{\lambda}_{(i)} - \widetilde{\delta}_{(i)}\right| = o_P(n^{-1}a_{np}^2) = o_P(1)\,.$$

A similar result does not hold for $\alpha \in (0,2)$: in this case $a_{np}^2/n \to \infty$. The condition $n^{-1}a_{np}^2 \to 0$ holds for sequences $p_n \to \infty$ satisfying $p_n = o(n^{(\alpha-2)/2})$. On the other hand, another condition on the growth of $(p_n)$ is required in (2.13). Taking into account both conditions, we need to require $p_n = O(n^\delta)$ for small values of $\delta > 0$.

**Remark 3.5.** Consider a random array $(h_{kl})$ independent of $(X_{it})$ and assume that the summability conditions (2.7) and (2.8) hold a.s. Then Theorem 3.1 remains valid conditionally on $(h_{kl})$, hence unconditionally in $P$-probability (see also [7]).

## 4. Some applications

The following result is a consequence of Theorem 3.1 and Lemma A.3. We write $\varepsilon_s$ for Dirac measure at $s$.

**Corollary 4.1.** *Assume $\alpha \in (0,2) \cup (2,4)$ and the conditions of Theorem 3.1 hold. Let $(E_i)$ be iid unit exponentials, $\Gamma_i = E_1 + \cdots + E_i$, $i \geq 1$. Then*

$$N_n = \sum_{i=1}^p \varepsilon_{a_{np}^{-2}\lambda_i} \xrightarrow{d} N = \sum_{i=1}^\infty \sum_{j=1}^r \varepsilon_{\Gamma_i^{-2/\alpha} v_j}\,, \tag{4.1}$$

*in the space of point measures with state space $\overline{\mathbb{R}}_0 = [\mathbb{R} \cup \{\pm\infty\}] \backslash \{0\}$ equipped with the vague topology.*

**Remark 4.2.** For $\alpha \in (0,2)$, the eigenvalues $(\lambda_i)$ of $\mathbf{X}_n\mathbf{X}_n'$ are non-negative and then one can choose the state space $(0,\infty)$ for the point processes $N_n$ and $N$. For $\alpha \in (2,4)$, the eigenvalues $(\lambda_i)$ of $\mathbf{X}_n\mathbf{X}_n' - E\mathbf{X}_n\mathbf{X}_n$ are not necessarily non-negative and therefore we have chosen the state space $\overline{\mathbb{R}}_0$. However, the corollary shows that all limiting points are positive.

*Proof.* We start with $\alpha \in (0,2)$. It follows from Lemma A.3 and Theorem A.1 that $(\sum_{i=1}^p \varepsilon_{a_{np}^{-2} D_i})$ converges in distribution to a Poisson random measure on $(0,\infty)$ with intensity measure of $(x,\infty)$, $x > 0$, given as the limit of (notice that $p_+ = 1$ in this case and $\alpha$ corresponds to $\alpha/2 \in (0,1)$)

$$pP(a_{np}^{-2} D > x) \sim p\, n\, P(Z^2 > a_{np}^2 x) \to x^{-\alpha/2}\,, \quad x > 0\,.$$

This means that

$$\sum_{i=1}^p \varepsilon_{a_{np}^{-2} D_i} \xrightarrow{d} \sum_{i=1}^\infty \varepsilon_{\Gamma_i^{-2/\alpha}}\,, \quad n \to \infty\,. \tag{4.2}$$

The continuous mapping theorem implies that

$$\sum_{j=1}^r \sum_{i=1}^p \varepsilon_{a_{np}^{-2} D_i v_j} \xrightarrow{d} \sum_{j=1}^r \sum_{i=1}^\infty \varepsilon_{\Gamma_i^{-2/\alpha} v_j}\,, \tag{4.3}$$

at least if $r$ is finite. However, this convergence can be extended from a finite positive integer $r^*$ to $r = \infty$ using a triangular convergence argument. In particular, it suffices to show that for any continuous function $f$ with compact support on $\overline{\mathbb{R}}_0$,

$$\lim_{r^* \to \infty} \limsup_n P\Big(\sum_{j=r^*+1}^\infty \sum_{i=1}^p |f(a_{np}^{-2} D_i v_j)| > \epsilon\Big) = 0 \tag{4.4}$$

and

$$\lim_{r^* \to \infty} P\Big(\sum_{j=r^*+1}^\infty \sum_{i=1}^p |f(\Gamma_i^{-2/\alpha} v_j)| > \epsilon\Big) = 0\,. \tag{4.5}$$

Since $f$ has compact support (only nonzero on sets bounded away from zero), there exists $\delta > 0$ such that $f(x) = 0$ for all $|x| \leq \delta$. It follows that the summand in (4.4) is positive if and only if the largest point among $\{a_{np}^{-2} D_i v_j, i = 1, \ldots, p, \ j = r^* + 1, r^* + 2, \ldots\}$ is greater than $\delta$. But the largest point is $a_{np}^{-2} D_{(1)} v_{r^*+1}$ and the limit of the probability this point exceeds $\delta$ is (see (4.2))

$$\begin{aligned} \lim_{n\to\infty} P(a_{np}^{-2} D_{(1)} v_{r^*+1} > \delta) &= 1 - \exp\{-(\delta/v_{r^*+1})^{-\alpha/2}\} \\ &\to 0, \text{ as } r^* \to \infty \,. \end{aligned}$$

Similarly for (4.5), the limit of the probability is bounded by

$$\lim_{r^*\to\infty} P(\Gamma_1^{-2/\alpha} v_{r^*+1}| > \delta) = \lim_{r^*\to\infty} \Big(1 - \exp\{-(\delta/v_{r^*+1})^{-\alpha/2}\}\Big) = 0 \,,$$

and hence (4.3) holds also in the case $r = \infty$. Now an application of (3.1) shows that the point process convergence remains valid with the points $(a_{np}^{-2} D_i v_j)$ replaced by $(a_{np}^{-2} \lambda_i)$.

For $\alpha \in (2, 4)$, the same argument shows that

$$\sum_{i=1}^{p} \varepsilon_{a_{np}^{-2}(D_i - ED)} \xrightarrow{d} \sum_{i=1}^{\infty} \varepsilon_{\Gamma_i^{-2/\alpha}}$$

and the limit is a Poisson random measure on $\overline{\mathbb{R}}_0$ with intensity measure of $(x, \infty)$, $x > 0$, given as the limit of

$$pP(a_{np}^{-2}(D - ED) > x) \sim p\, n\, P(Z^2 - EZ^2 > a_{np}^2 x) \to (x^{-1})^{\alpha/2} \,, \quad x > 0 \,,$$

while the intensity measure on $(-\infty, -x]$, $x > 0$, vanishes:

$$pP(a_{np}^{-2}(D - ED) \le -x) \sim p\, n\, P(Z^2 - EZ^2 \le -a_{np}^2 x) \to 0 \,, \quad x > 0 \,.$$

The continuous mapping theorem yields

$$\sum_{j=1}^{r} \sum_{i=1}^{p} \varepsilon_{a_{np}^{-2}(D_i - ED) v_j} \xrightarrow{d} \sum_{j=1}^{r} \sum_{i=1}^{\infty} \varepsilon_{\Gamma_i^{-2/\alpha} v_j} \,.$$

An application of (3.2) shows that the point process convergence remains valid with the points $(a_{np}^{-2}(D_i - ED) v_j)$ replaced by $(a_{np}^{-2} \lambda_i)$. □

Corollary 4.1 and the continuous mapping theorem immediately yield results about the joint convergence of the largest eigenvalues of the matrices $\mathbf{X}_n \mathbf{X}_n'$ for $\alpha \in (0, 2)$ and of the matrices $\mathbf{X}_n \mathbf{X}_n' - E\mathbf{X}_n \mathbf{X}_n'$ for $\alpha \in (2, 4)$. In what follows, we write $\lambda_{(p)} \le \cdots \le \lambda_{(1)}$ for the order statistics of the eigenvalues $(\lambda_i)$ in both situations. An application of (4.1) then yields for every fixed $k \ge 1$,

$$a_{np}^{-2} \big(\lambda_{(1)}, \ldots, \lambda_{(k)}\big) \xrightarrow{d} \big(d_{(1)}, \ldots, d_{(k)}\big) \,,$$

where $d_{(1)} \ge \cdots \ge d_{(k)}$ are the $k$ largest ordered values of the set $\{\Gamma_i^{-2/\alpha} v_j, i = 1, 2, \ldots, j = 1, \ldots, r\}$. The continuous mapping theorem yields for $k \ge 1$,

$$\frac{\lambda_{(1)}}{\lambda_{(1)} + \cdots + \lambda_{(k)}} \xrightarrow{d} \frac{d_{(1)}}{d_{(1)} + \cdots + d_{(k)}} \,, \quad n \to \infty \,. \tag{4.6}$$

An application of the continuous mapping theorem to (4.1) in the spirit of Resnick [16], Theorem 7.1, also yields the following result.

**Corollary 4.3.** *Assume the condition of Theorem 3.1. Then the following limit results hold.*

(1) *If $\alpha \in (0, 2)$ then*

$$a_{np}^{-2} \Big(\lambda_{(1)}, \sum_{i=1}^{p} \lambda_i\Big) \quad \xrightarrow{d} \quad \Big(v_1\, \Gamma_1^{-2/\alpha}, \sum_{j=1}^{r} v_j \sum_{i=1}^{\infty} \Gamma_i^{-2/\alpha}\Big) \,,$$

*where $\Gamma_1^{-2/\alpha}$ is Fréchet distributed with distribution function $\Phi_{\alpha/2} = \mathrm{e}^{-x^{-\alpha/2}}$, $x>0$, and $\sum_{i=1}^\infty \Gamma_i^{-2/\alpha}$ has the distribution of a positive $\alpha/2$-stable random variable. In particular,*

$$\frac{\lambda_{(1)}}{\lambda_1+\cdots+\lambda_p} \xrightarrow{d} \frac{v_1}{\sum_{j=1}^r v_j}\,\frac{\Gamma_1^{-2/\alpha}}{\sum_{i=1}^\infty \Gamma_i^{-2/\alpha}}\,,\quad n\to\infty\,. \tag{4.7}$$

(2) *If $\alpha\in(2,4)$ then*

$$a_{np}^{-2}\Big(\lambda_{(1)},\sum_{i=1}^p\lambda_i\Big) \;\xrightarrow{d}\; \Big(v_1\Gamma_1^{-2/\alpha},\sum_{j=1}^r v_j\lim_{\gamma\downarrow 0}\sum_{i=1}^\infty\Big(\Gamma_i^{-2/\alpha} I_{\{\Gamma_i^{-2/\alpha}>\gamma\}} - E\Gamma_i^{-2/\alpha} I_{\{\Gamma_i^{-2/\alpha}>\gamma\}}\Big)\Big)\,,$$

*where $\Gamma_1^{-2/\alpha}$ is Fréchet $\Phi_{\alpha/2}$ distributed and the second component is an $\alpha/2$-stable random variable $\xi_{\alpha/2}$ which is totally skewed to the right. In particular,*

$$\frac{\lambda_{(1)}}{\lambda_1+\cdots+\lambda_p} \xrightarrow{d} \frac{v_1}{\sum_{j=1}^r v_j}\,\frac{\Gamma_1^{-2/\alpha}}{\xi_{\alpha/2}}\,,\quad n\to\infty\,. \tag{4.8}$$

**Remark 4.4.** The fact that $\sum_{i=1}^\infty \Gamma_i^{-2/\alpha}$ has an $\alpha/2$-stable distribution for $\alpha\in(0,2)$ can be found e.g. in Resnick [16], Theorem 7.1, or in Samorodnitsky and Taqqu [17]. For $\alpha\in(2,4)$, the same references can be used to detect that the limit $\xi_{\alpha/2}$ exists a.s. and represents an $\alpha/2$-stable limit which is totally skewed to the right.

**Remark 4.5.** It follows from Corollary 4.3 that for fixed $k\ge 1$,

$$\frac{\lambda_{(1)}+\cdots\lambda_{(k)}}{\lambda_1+\cdots+\lambda_p} \xrightarrow{d} \frac{d_{(1)}+\cdots+d_{(k)}}{d_{(1)}+d_{(2)}+\cdots}\,,$$

Unfortunately, the limiting variable does not in general have a clean form. An exception is the case when $r=1$; see e.g. Example 4.7.

*Proof.* We only give the argument for $\alpha\in(0,2)$; the case $\alpha\in(2,4)$ is similar and therefore omitted. We restrict ourselves to a sketch of the proof of the convergence of the sum of the eigenvalues, i.e. to the convergence of the trace of $a_{np}^{-2}\mathbf{X}_n\mathbf{X}_n'$; the joint convergence with $a_{np}^{-2}\lambda_{(1)}$ follows along the lines of the proof of Theorem 3.1 by observing the fact that we only exploit the convergence of the sums $a_{np}^{-2}\sum_{l=0}^\infty\sum_{k=0}^\infty h_{kl}^2\sum_{t=1}^n\sum_{i=1}^p Z_{i-k,t-l}^2$. By using the equality of the traces, we have

$$\begin{aligned}\operatorname{tr}(\mathbf{X}_n\mathbf{X}_n') = \sum_{i=1}^p\lambda_i &= \sum_{i=1}^p\sum_{t=1}^n X_{it}^2\\ &= \sum_{i=1}^p\sum_{t=1}^n\Big[\sum_{l=0}^\infty\sum_{k=0}^\infty h_{kl}^2 Z_{i-k,t-l}^2 + R_{it}^{(n)}\Big]\\ &= A_1+A_2\,.\end{aligned}$$

The proof of $a_{np}^{-2}A_2\xrightarrow{P}0$ now follows along the lines of the proof of Lemma 5.1; it is actually much simpler since this time one does not have to take into account the operations $\max_{i=1,\dots,p}$. For $m\ge1$ fixed, we write

$$a_{np}^{-2}A_1 = a_{np}^{-2}\sum_{i=1}^p\sum_{t=1}^n\sum_{l\vee k\le m} h_{kl}^2 Z_{i-k,t-l}^2 + R_{n,m}\,.$$

Following an argument similar to that given for Lemma 5.1, we have

$$\lim_{m\to\infty}\limsup_{n\to\infty} P(|R_{n,m}|>\varepsilon)=0\,,\quad \varepsilon>0\,.$$

Set $S_{kl} = \sum_{t=1}^n \sum_{i=1}^p Z^2_{i-k,t-l}$ , and observe that for fixed $k, l$,

$$a_{np}^{-2}(S_{kl} - S_{00}) \xrightarrow{P} 0\,, \quad n \to \infty\,,$$

and (see the proof of Lemma 5.3)

$$a_{np}^{-2} S_{00} \xrightarrow{d} \sum_{i=1}^{\infty} \Gamma_i^{-2/\alpha} = \xi_{\alpha/2}\,, \quad n \to \infty\,,$$

where $\xi_{\alpha/2}$ has an $\alpha/2$-stable distribution. Then it follows that

$$\begin{aligned}\sum_{k\vee l\le m} h^2_{kl} S_{kl} &\xrightarrow{d} \sum_{k\vee l\le m} h^2_{kl}\, \xi_{\alpha/2}\,, \quad n \to \infty\,,\\ &\xrightarrow{\text{a.s.}} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} h^2_{kl}\, \xi_{\alpha/2}\,, \quad m \to \infty\,.\end{aligned}$$

Finally, we observe that (2.3) holds. A combination of the arguments above concludes the proof in the case $\alpha \in (0, 2)$. □

To illustrate the theory we consider a simple moving average example.

**Example 4.6.** Assume that $\alpha \in (0, 2)$ and

$$X_{it} = Z_{it} - Z_{i,t-1} - 2(Z_{i-1,t} - Z_{i-1,t-1})\,, \quad i, t \in \mathbb{Z}\,. \tag{4.9}$$

In this case, $\mathbf{h}_0 = (1, -1, 0, 0, \ldots)'$, $\mathbf{h}_1 = (-2, 2, 0, 0, \ldots)'$ and hence $M_2 = \operatorname{diag}(2, 8)$ which has the positive eigenvalues $v_1 = 8$ and $v_2 = 2$. The limit point process in (4.1) is

$$N = \sum_{i=1}^{\infty} \varepsilon_{8\Gamma_i^{-2/\alpha}} + \sum_{i=1}^{\infty} \varepsilon_{2\Gamma_i^{-2/\alpha}}\,,$$

so that

$$a_{np}^{-2}\left(\lambda_{(1)}, \lambda_{(2)}\right) \xrightarrow{d} \left(8\Gamma_1^{-2/\alpha}, 2\Gamma_1^{-2/\alpha} \vee 8\Gamma_2^{-2/\alpha}\right),$$

In particular, we have for the normalized spectral gap

$$a_{np}^{-2}\big(\lambda_{(1)} - \lambda_{(2)}\big) \xrightarrow{d} 6\Gamma_1^{-2/\alpha} I_{\{\Gamma_1 4^{\alpha/2} < \Gamma_2\}} + 8\big(\Gamma_1^{-2/\alpha} - \Gamma_2^{-2/\alpha}\big) I_{\{\Gamma_1 4^{\alpha/2} > \Gamma_2\}}$$

and for the self-normalized spectral gap

$$\begin{aligned}\frac{\lambda_{(1)} - \lambda_{(2)}}{\lambda_{(1)}} &\xrightarrow{d} \frac{6}{8} I_{\{\Gamma_1 2^\alpha < \Gamma_2\}} + \big(1 - (\Gamma_1/\Gamma_2)^{2/\alpha}\big) I_{\{\Gamma_1 2^\alpha > \Gamma_2\}}\\ &= \frac{3}{4} I_{\{U 2^\alpha < 1\}} + \big(1 - U^{2/\alpha}\big) I_{\{U 2^\alpha > 1\}}\,,\end{aligned}$$

for a uniform random variable $U = \Gamma_1/\Gamma_2$ on $(0, 1)$. The limit distribution of the spectral gap has an atom at $3/4$ with probability $2^{-\alpha}$. Along these same lines, we also have

$$(a_{np}^{-2}\lambda_{(1)}, \lambda_{(2)}/\lambda_{(1)}) \xrightarrow{d} (8\Gamma_1^{-2/\alpha}, \frac{1}{4} I_{\{U<2^{-\alpha}\}} + U^{2/\alpha} I_{\{U \ge 2^{-\alpha}\}})$$

and hence the limit distribution of $\lambda_{(2)}/\lambda_{(1)}$ is supported on $[1/4, 1)$ with mass of $2^{-\alpha}$ at $1/4$. The histogram of the ratio $\big(\lambda_{(2)}/\lambda_{(1)}\big)^{2/\alpha}$ based on 1000 replications from the model (4.9) with noise given by a $t$-distribution with $\alpha = 1.5$ degrees of freedom, $n = 1000$ and $p = 200$ is displayed in Figure 1. It is evident that the histogram is remarkably close to what one would expect from a sample from a the truncated uniform, $.3536 I_{\{U<.3536\}} + U_{\{U\ge .3536\}}$. The mass of the limiting discrete

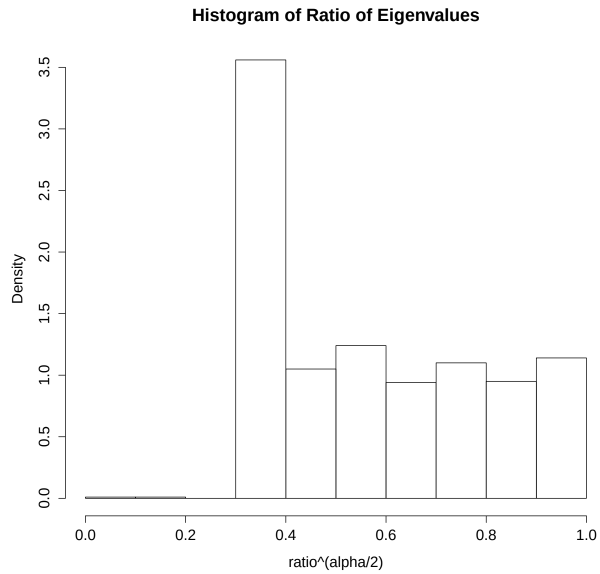


FIGURE 1. Histogram based on 1000 replications of $\left(\lambda_{(2)}/\lambda_{(1)}\right)^{2/\alpha}$ from model (4.9).

component of the ratio can be much larger if one conditions on $a_{np}^{-2}\lambda_{(1)}$ being large. Specifically, for any $\epsilon \in (0, 1/4)$ and $x > 0$,

$$\lim_{n\to\infty} P(\epsilon < \lambda_{(2)}/\lambda_{(1)} \le 1/4|\lambda_{(1)} > a_{np}^2 x) = P(\Gamma_1/\Gamma_2 \le 2^{-\alpha}|\Gamma_1 < (x/8)^{-\alpha/2}) =: H(x)\,.$$

The function $H$ approaches 1 as $x \to \infty$ indicating the speed at which the two largest eigenvalues become linearly related (see Figure 2 for a graph of $H$ in the case $\alpha = 1.5$). In addition, from

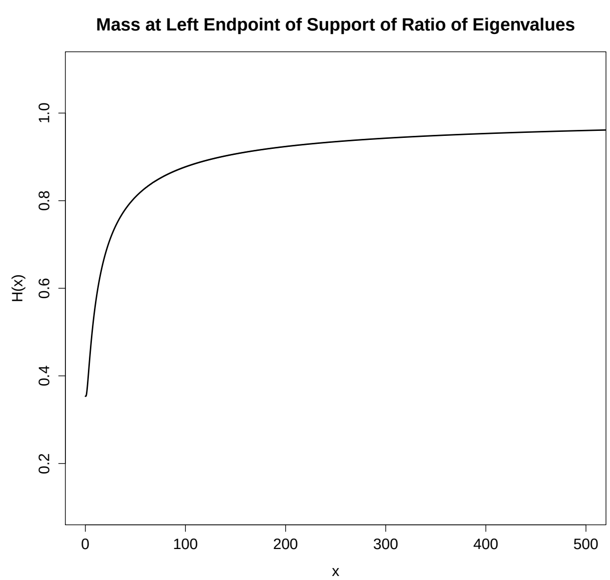


FIGURE 2. Graph of $H(x) = P(\Gamma_1/\Gamma_2 \le 2^{-\alpha}|\Gamma_1 < (x/8)^{-\alpha/2})$ when $\alpha = 1.5$.

Remark 4.5, we also have

$$\frac{\lambda_{(1)}}{\lambda_1+\cdots+\lambda_p} \stackrel{d}{\to} \frac{4}{5}\frac{\Gamma_1^{-2/\alpha}}{\sum_{i=1}^\infty \Gamma_i^{-2/\alpha}}\,.$$

Clearly, the limit random variable is stochastically smaller than what one would get in the iid case.

This example is essentially a prototype for the behavior of the extreme eigenvalues in the case when the rank $r$ of the matrix $M$ is larger than one. For example, the limiting distribution of the ratio of $\lambda_{(2)}/\lambda_{(1)}$ is

$$\lambda_{(2)}/\lambda_{(1)} \stackrel{d}{\to} \frac{v_2}{v_1} I_{\{U<(v_2/v_1)^{\alpha/2}\}} + U^{2/\alpha} I_{\{U\geq (v_2/v_1)^{\alpha/2}\}})\,,$$

which has limiting support on $[v_2/v_1,1)$ with mass $(v_2/v_1)^{\alpha/2}$ at the left-hand endpoint. So unlike the independent row case, the limiting distribution of the ratio of the smallest to the largest eigenvalues can have support bounded away from 0 with non-zero mass at the left-endpoint.

**Example 4.7.** Consider the separable case, i.e. $h_{kl}=\theta_k c_l$, $l\geq 0$, where $(c_l)$, $(\theta_k)$ are real sequences such that the conditions on $(h_{kl})$ in Theorem 3.1 hold. In this case,

$$M=\sum_{l=0}^\infty c_l^2\;(\theta_i\theta_j)_{i,j\geq 0}\,.$$

Note that $r=1$ with the only non-negative eigenvalue $v_1=\sum_{l=0}^\infty c_l^2\sum_{k=0}^\infty \theta_k^2$. In this case, the limit point process in Corollary 4.1 is a Poisson process on $(0,\infty)$ with mean measure of $(y,\infty)$ given by $(v_1/y)^{\alpha/2}$, $y>0$. This means that the point process of the normalized eigenvalues $(a_{np}^{-2}\lambda_{(i)})_{i=1,\ldots,p}$ has the same asymptotic behavior as the point process of the normalized points $(a_{np}^{-2}Z_{it})_{t=1,\ldots,n,i=1,\ldots,p}$.

## 5. Proof of Theorem 3.1

In what follows, we will use the matrix norms $\|A\|_2$ and $\|A\|_\infty$ for any $p\times p$ matrix $A=(A_{ij})$, i.e. $\|A\|_2$ is the square root of the largest eigenvalue of the matrix $AA'$ and $\|A\|_\infty=\max_{i=1,\ldots,p}\sum_{j=1}^p |A_{ij}|$. We will frequently make use of the bound $\|A\|_2\leq\|A\|_\infty$.

We break the proof of Theorem 3.1 into four steps via a series of lemmas.

**Step 1: Approximation of $\mathbf{X}_n\mathbf{X}_n'$ by $\mathbf{X}_n^{(m)}(\mathbf{X}_n^{(m)})'$.** First, we truncate the infinite series $X_{it}$ in (1.1) and show that it suffices to deal with the finite moving averages

$$X_{it}^{(m)}=\sum_{l=0}^\infty\sum_{k=0}^m h_{kl}Z_{i-k,t-l}\,,\quad m\geq 1\,.$$

and the corresponding matrices $\mathbf{X}_n^{(m)}=(X_{it}^{(m)})_{i=1,\ldots,p,t=1,\ldots,n}$.

**Lemma 5.1.** *Assume the conditions of Theorem 3.1. If $\alpha\in(0,1]$ and $E|Z|=\infty$ or if $\alpha\in[1,2)$ and $EZ=0$, then*

$$\lim_{m\to\infty}\limsup_{n\to\infty} P(a_{np}^{-2}\|\mathbf{X}_n\mathbf{X}_n'-\mathbf{X}_n^{(m)}(\mathbf{X}_n^{(m)})'\|_2>\varepsilon)=0\,,\quad \varepsilon>0\,.$$

*If $\alpha\in(2,4)$ and $EZ=0$, then*

$$\lim_{m\to\infty}\limsup_{n\to\infty} P(a_{np}^{-2}\|[\mathbf{X}_n\mathbf{X}_n'-E\mathbf{X}_n\mathbf{X}_n']-[(\mathbf{X}_n^{(m)})(\mathbf{X}_n^{(m)})'-E(\mathbf{X}_n^{(m)}(\mathbf{X}_n^{(m)})']\|_2>\varepsilon)=0\,,\quad\varepsilon>0\,.$$

*Proof.* **(1) The case $\alpha \in (0,2)$.** Observe that

$$\Big(\mathbf{X}_n\mathbf{X}_n' - \mathbf{X}_n^{(m)}(\mathbf{X}_n^{(m)})'\Big)_{ij} = \sum_{l_1,l_2=0}^{\infty} \sum_{k_1 \vee k_2 > m}^{\infty} h_{k_1,l_1} h_{k_2,l_2} \sum_{t=1}^{n} Z_{i-k_1,t-l_1} Z_{j-k_2,t-l_2}\,.$$

Since $\|A\|_2 \le \|A\|_\infty$ we have for $\alpha \in (0,2)$,

$$\begin{aligned}
& a_{np}^{-2}\|\mathbf{X}_n\mathbf{X}_n' - \mathbf{X}_n^{(m)}(\mathbf{X}_n^{(m)})'\|_2 \\
\le\ & a_{np}^{-2} \max_{i=1,\ldots,p} \sum_{j=1}^{p} \Big| \sum_{l_1,l_2=0}^{\infty} \sum_{k_1\vee k_2>m} h_{k_1,l_1} h_{k_2,l_2} \sum_{t=1}^{n} Z_{i-k_1,t-l_1} Z_{j-k_2,t-l_2} \Big| \\
\le\ & a_{np}^{-2} \max_{i=1,\ldots,p} \sum_{j=1}^{p} \Big| \sum_{l=0}^{\infty} \sum_{k\vee(k+j-i)>m} h_{kl} h_{k+j-i,l} \sum_{t=1}^{n} Z_{i-k,t-l}^2 \Big| \\
& + a_{np}^{-2} \max_{i=1,\ldots,p} \sum_{j=1}^{p} \Big| \sum_{l_1,l_2=0,l_1\ne l_2}^{\infty} \sum_{k_1\vee k_2>m} h_{k_1,l_1} h_{k_2,l_2} \sum_{t=1}^{n} Z_{i-k_1,t-l_1} Z_{j-k_2,t-l_2} \Big| \\
& + a_{np}^{-2} \max_{i=1,\ldots,p} \sum_{j=1}^{p} \Big| \sum_{l=0}^{\infty} \sum_{k_1\vee k_2>m, i-k_1\ne j-k_2} h_{k_1,l} h_{k_2,l} \sum_{t=1}^{n} Z_{i-k_1,t-l} Z_{j-k_2,t-l} \Big| \\
=\ & I_n^{(1)} + I_n^{(2)} + I_n^{(3)}\,.
\end{aligned} \tag{5.1}$$

*Bounds for $P(I_n^{(1)} > \varepsilon)$.* We have

$$\begin{aligned}
I_n^{(1)} \le\ & a_{np}^{-2} \max_{i=1,\ldots,p} \sum_{j=1}^{p} \sum_{l=0}^{\infty} \sum_{k>m,k+j-i\ge 0} |h_{kl} h_{k+j-i,l}| \sum_{t=1}^{n} Z_{i-k,t-l}^2 \\
& + a_{np}^{-2} \max_{i=1,\ldots,p} \sum_{j=1}^{p} \sum_{l=0}^{\infty} \sum_{k\le m\,,k+j-i>m} |h_{kl} h_{k+j-i,l}| \sum_{t=1}^{n} Z_{i-k,t-l}^2 \\
\le\ & c\, a_{np}^{-2} \max_{i=1,\ldots,p} \sum_{l=0}^{\infty} \sum_{k=m+1}^{\infty} |h_{kl}| \sum_{t=1}^{n} Z_{i-k,t-l}^2 \\
& + c\, a_{np}^{-2} \max_{i=1,\ldots,p} \sum_{s=m+1}^{\infty} \sum_{l=0}^{\infty} \sum_{k=1}^{m} |h_{sl} h_{kl}| \sum_{t=1}^{n} Z_{i-k,t-l}^2 \\
=\ & I_n^{(11)} + I_n^{(12)}\,.
\end{aligned}$$

Then for $\varepsilon > 0$ and a sequence $(Z_t)$ of iid copies of $Z$,

$$\begin{aligned}
P(I_n^{(11)} > \varepsilon) &\leq p\,P\Big(c\sum_{l=0}^{\infty}\sum_{k=m+1}^{\infty}|h_{kl}|\sum_{t=1}^{n} Z_{k,t-l}^2 > \varepsilon a_{np}^2\Big)\\
&\leq p\,P\Big(c\sum_{l=0}^{\infty}\sum_{k=m+1}^{\infty}|h_{kl}|\sum_{t=1}^{n} Z_{k,t-l}^2 I_{\{|h_{kl}|\sum_{s=1}^n Z_{k,s-l}^2 > a_{np}^2\}} > \varepsilon a_{np}^2\Big)\\
&\quad + p\,P\Big(c\sum_{l=0}^{\infty}\sum_{k=m+1}^{\infty}|h_{kl}|\sum_{t=1}^{n} Z_{k,t-l}^2 I_{\{|h_{kl}|\sum_{s=1}^n Z_{k,s-l}^2 \leq a_{np}^2\}} > \varepsilon a_{np}^2\Big)\\
&\leq p\,c\sum_{l=0}^{\infty}\sum_{k=m+1}^{\infty} P\Big(|h_{kl}|\sum_{t=1}^{n} Z_t^2 > a_{np}^2\Big)\\
&\quad + p\,c\sum_{l=0}^{\infty}\sum_{k=m+1}^{\infty} E\Big[a_{np}^{-2}|h_{kl}|\sum_{t=1}^{n} Z_t^2 I_{\{|h_{kl}|\sum_{s=1}^n Z_s^2 \leq a_{np}^2\}}\Big]\\
&= P_n^{(1)} + P_n^{(2)}\,.
\end{aligned}$$

In view of the uniform large deviation result of Theorem A.1 we have

$$P_n^{(1)} \leq c\,pnP(Z^2 > a_{np}^2)\sum_{l=0}^{\infty}\sum_{k=m+1}^{\infty}|h_{kl}|^{\alpha/2} \leq c\sum_{l=0}^{\infty}\sum_{k=m+1}^{\infty}|h_{kl}|^{\alpha/2}\,,$$

for a constant $c$ which does not depend on $m$, and the right-hand side converges to zero as $m \to \infty$. For $P_n^{(2)}$, we observe that, by the Karamata and uniform convergence theorems for regularly varying functions,

$$\begin{aligned}
P_n^{(2)} &\leq p\,c\sum_{l=0}^{\infty}\sum_{k=m+1}^{\infty} E\Big[a_{np}^{-2}|h_{kl}|\sum_{t=1}^{n} Z_t^2 I_{\{|h_{kl}|\,Z_t^2 \leq a_{np}^2\}}\Big]\\
&= pn\,c\sum_{l=0}^{\infty}\sum_{k=m+1}^{\infty} E\Big[a_{np}^{-2}|h_{kl}|\,Z^2 I_{\{|h_{kl}|\,Z^2 \leq a_{np}^2\}}\Big]\\
&\leq pnc\sum_{l=0}^{\infty}\sum_{k=m+1}^{\infty} P(|h_{kl}|Z^2 > a_{np}^2)\\
&\leq c\sum_{l=0}^{\infty}\sum_{k=m+1}^{\infty}|h_{kl}|^{\alpha/2}\,.
\end{aligned}$$

This is the same bound as for $P_n^{(1)}$. This proves that

$$\lim_{m\to\infty}\limsup_{n\to\infty} P(I_n^{(11)} > \varepsilon) = 0\,, \quad \varepsilon > 0\,.$$

For $I_n^{(12)}$, we can proceed in a similar way. Observe that for $\varepsilon > 0$,

$$\begin{aligned}
P(I_n^{(12)} > \varepsilon) &\leq pP\Big(c\, a_{np}^{-2} \sum_{s=m+1}^{\infty} \sum_{l=0}^{\infty} \sum_{k=1}^{m} |h_{sl} h_{kl}| \sum_{t=1}^{n} Z_{k,t-l}^2 > \varepsilon\Big) \\
&\leq p \sum_{s=m+1}^{\infty} \sum_{l=0}^{\infty} \sum_{k=1}^{m} P\Big(|h_{sl} h_{kl}| \sum_{t=1}^{n} Z_t^2 > a_{np}^2\Big) \\
&\quad + p \sum_{s=m+1}^{\infty} \sum_{l=0}^{\infty} \sum_{k=1}^{m} E\Big[a_{np}^{-2} |h_{sl} h_{kl}| \sum_{t=1}^{n} Z_t^2 I_{\{|h_{sl} h_{kl}| \sum_{s=1}^{n} Z_s^2 \leq a_{np}^2\}}\Big] \\
&\leq np\, c \sum_{s=m+1}^{\infty} \sum_{l=0}^{\infty} \sum_{k=1}^{m} P(|h_{sl} h_{kl}| Z^2 > a_{np}^2) \\
&\leq c \sum_{s=m+1}^{\infty} \sum_{l=0}^{\infty} \sum_{k=1}^{m} |h_{sl} h_{kl}|^{\alpha/2} \leq c \sum_{s=m+1}^{\infty} \sum_{l=0}^{\infty} |h_{sl}|^{\alpha/2},
\end{aligned}$$

and the right-hand side converges to zero as $m \to \infty$. This completes the proof of

$$\lim_{m\to\infty} \limsup_{n\to\infty} P(I_n^{(1)} > \varepsilon) = 0, \quad \varepsilon > 0.$$

*Bounds for* $P(I_n^{(2)} > \varepsilon)$. Consider the index set

$$S = \{j = 1, \ldots, p; 0 \leq l_1 \neq l_2 < \infty; k_1 \vee k_2 > m\}.$$

We observe for $\varepsilon > 0$ that

$$\begin{aligned}
& P(I_n^{(2)} > \varepsilon) \\
\leq\ & p\, P\Big(\sum_S |h_{k_1,l_1} h_{k_2,l_2}| \Big| \sum_{t=1}^{n} Z_{i-k_1,t-l_1} Z_{j-k_2,t-l_2}\Big| > \varepsilon a_{np}^2\Big) \\
\leq\ & p\, P\Big(\sum_S |h_{k_1,l_1} h_{k_2,l_2}| \times \\
& \Big|\sum_{t=1}^{n} Z_{i-k_1,t-l_1} Z_{j-k_2,t-l_2} I_{\{|h_{k_1,l_1} h_{k_2,l_2}||Z_{i-k_1,t-l_1} Z_{j-k_2,t-l_2}| > a_{np}^2\}}\Big| > \varepsilon a_{np}^2\Big) \\
& + p\, P\Big(\sum_S |h_{k_1,l_1} h_{k_2,l_2}| \times \\
& \Big|\sum_{t=1}^{n} Z_{i-k_1,t-l_1} Z_{j-k_2,t-l_2} I_{\{|h_{k_1,l_1} h_{k_2,l_2}||Z_{i-k_1,t-l_1} Z_{j-k_2,t-l_2}| \leq a_{np}^2\}}\Big| > \varepsilon a_{np}^2\Big) \\
=\ & P_n^{(3)} + P_n^{(4)}.
\end{aligned}$$

We start with the case $\alpha < 1$. Then, by the uniform convergence theorem for regularly varying functions,

$$\begin{aligned}
P_n^{(3)} &\leq pn \sum_S P\Big(|h_{k_1,l_1} h_{k_2,l_2}||Z_1 Z_2| > a_{np}^2\Big) \\
&\leq cp[npP(|Z_1 Z_2| > a_{np}^2)]\Big[p^{-1} \sum_S |h_{k_1,l_1} h_{k_2,l_2}|^{\alpha}\Big] \to 0, \quad n \to \infty, \qquad (5.2)
\end{aligned}$$

using that $P(|Z_1Z_2|>x)=x^{-\alpha}L_2(x)$ for a slowly varying function $L_2$, the fact that

$$p^{-1}\sum_S |h_{k_1,l_1}h_{k_2,l_2}|^\alpha = \sum_{0\le l_1\neq l_2<\infty\,;k_1\vee k_2>m} |h_{k_1,l_1}h_{k_2,l_2}|^\alpha \le \Big(\sum_{l=0}^\infty\sum_{k=0}^\infty |h_{kl}|^\alpha\Big)^2<\infty$$

and condition (2.10). On the other hand, by Karamata's integral theorem, as $n\to\infty$.

$$\begin{aligned} P_n^{(4)} &\le pn\sum_S E[a_{np}^{-2}|h_{k_1,l_1}h_{k_2,l_2}||Z_1Z_2|\,I_{\{|h_{k_1,l_1}h_{k_2,l_2}||Z_1Z_2|\le a_{np}^2\}}] \\ &\le cpn\sum_S P\Big(|h_{k_1,l_1}h_{k_2,l_2}||Z_1Z_2|>a_{np}^2\Big)\to 0\,. \end{aligned} \tag{5.3}$$

Next we consider the case $\alpha=1$ and $E|Z|=\infty$. The term $P_n^{(3)}$ can be bounded by (5.2) as for $\alpha<1$ while the function

$$L_3(x)=\frac{E|x^{-1}Z_1Z_2|I_{\{|Z_1Z_2|\le x\}}}{P(|Z_1Z_2|>x)}\,,\quad x>0\,,$$

is slowly varying and converges to infinity (Bingham et al. [4], (1.5.8)). But

$$E|x^{-1}Z_1Z_2|I_{\{|Z_1Z_2|\le x\}}+P(|Z_1Z_2|>x)\,,\quad x>0,$$

decreases and is regularly varying with index $-1$. Therefore, using the uniform convergence theorem for decreasing regularly varying functions, the expression on the right-hand side of (5.3) can be bounded by

$$cp[npEa_{np}^{-2}|Z_1Z_2|I_{\{|Z_1Z_2|\le a_{np}^2\}}]\Big[p^{-1}\sum_S |h_{k_1,l_1}h_{k_2,l_2}|\Big]\,.$$

The same argument as for $\alpha<1$ applies, expect that we have to use (2.11) instead of (2.10).

Now we turn to the case $\alpha\in(1,2)$ and $EZ=0$. Again, $P_n^{(3)}$ can be bounded in the same way as for $\alpha<1$, observing that condition (2.12) is stronger than (2.10). For $P_n^{(4)}$ we have the following bound. Choose some $\gamma\in(\alpha,2]$. Then, using the Markov and Hölder inequalities,

$$\begin{aligned} P_n^{(4)} &\le c\,pa_{np}^{-2\gamma}E\Big|\sum_S |h_{k_1,l_1}h_{k_2,l_2}|\,\Big|\sum_{t=1}^n Z_{1-k_1,t-l_1}Z_{j-k_2,t-l_2}I_{\{|h_{k_1,l_1}h_{k_2,l_2}|\,|Z_{1-k_1,t-l_1}Z_{j-k_2,t-l_2}|\le a_{np}^2\}}\Big|\Big|^\gamma \\ &\le c\,pa_{np}^{-2\gamma}\sum_S |h_{k_1,l_1}h_{k_2,l_2}|E\Big|\sum_{t=1}^n Z_{1-k_1,t-l_1}Z_{j-k_2,t-l_2}I_{\{|h_{k_1,l_1}h_{k_2,l_2}|\,|Z_{1-k_1,t-l_1}Z_{j-k_2,t-l_2}|\le a_{np}^2\}}\Big|^\gamma \\ &\quad\times\Big(\sum_S |h_{k_1,l_1}h_{k_2,l_2}|\Big)^{\gamma-1} \\ &\le c\,p^\gamma a_{np}^{-2\gamma}\sum_S |h_{k_1,l_1}h_{k_2,l_2}|E\Big|\sum_{t=1}^n Z_{1-k_1,t-l_1}Z_{j-k_2,t-l_2}I_{\{|h_{k_1,l_1}h_{k_2,l_2}|\,|Z_{1-k_1,t-l_1}Z_{j-k_2,t-l_2}|\le a_{np}^2\}} \\ &\quad -nEZ_1Z_2I_{\{|h_{k_1,l_1}h_{k_2,l_2}|\,|Z_1Z_2|\le a_{np}^2\}}\Big|^\gamma \\ &\quad +c\,n^\gamma p^\gamma a_{np}^{-2\gamma}\sum_S |h_{k_1,l_1}h_{k_2,l_2}|\Big(E|Z_1Z_2|I_{\{|h_{k_1,l_1}h_{k_2,l_2}|\,|Z_1Z_2|>a_{np}^2\}}\Big)^\gamma \\ &= P_n^{(41)}+P_n^{(42)}\,. \end{aligned}$$

In the last step we used $E(Z_1Z_2)=0$. Using Karamata's theorem and (2.12), we have

$$\begin{aligned} P_n^{(42)} &\le c\,n^\gamma p^\gamma\sum_{S'} |h_{k_1,l_1}h_{k_2,l_2}|^{1+\gamma(\alpha-1)}[P(|Z_1Z_2|>a_{np}^2)]^\gamma \\ &\le c\,[P(|Z_1Z_2|>a_{np}^2)]^\gamma n^\gamma p^{\gamma+1}\to 0\,,\quad n\to\infty\,. \end{aligned}$$

Inequality (A.3) in Lemma A.4 and Karamata's theorem yield, choosing $\gamma$ close to $\alpha$,

$$\begin{aligned}
P_n^{(41)} &\le c\,p^\gamma n a_{np}^{-2\gamma} \sum_{S'} |h_{k_1,l_1} h_{k_2,l_2}| E|Z_1Z_2|^\gamma I_{\{|h_{k_1,l_1} h_{k_2,l_2}|\,|Z_1Z_2|\le a_{np}^2\}} \\
&\le c\,np^\gamma \sum_{S'} |h_{k_1,l_1} h_{k_2,l_2}|^{1-\gamma+\alpha} P(|Z_1Z_2| > a_{np}^2) \\
&\le c\,n\,p^{\gamma+1} P(|Z_1Z_2| > a_{np}^2) \to 0\,, \quad n\to\infty\,.
\end{aligned}$$

In the last step we used (2.12).

In the case $\alpha = 1$ and $EZ = 0$ we can follow the lines of the proof for $\alpha \in (1,2)$ with one exception: the use of Karamata's theorem for bounding $P_n^{(42)}$. Karamata's theorem tells us that

$$L_4(x) = \frac{E|x^{-1}Z_1Z_2| I_{\{|Z_1Z_2|>x\}}}{P(|Z_1Z_2|>x)}$$

is a slowly varying function which converges to infinity as $x\to\infty$. In this case, we obtain for $\gamma > 1$ arbitrarily close to 1 and $\epsilon > 0$ arbitrarily close to zero and large $n$,

$$P_n^{(42)} \le cp\,[np\,P(|Z_1Z_2| > a_{np}^2)(np)^\epsilon]^\gamma \le c\,p\,(np)^{\gamma(-1+2\epsilon)} \to 0,$$

if we choose $\gamma(-1+2\epsilon) < -1$.

Thus we proved that

$$\lim_{m\to\infty} \limsup_{n\to\infty} P(I_n^{(i)} > \varepsilon) = 0\,, \quad i = 1,2.$$

The proof for $i = 3$ is analogous to $i = 2$ and therefore omitted. The proof of the lemma is complete in the case $\alpha \in (0,2)$.

**(2) The case $\alpha \in (2,4)$.** We start by observing that (5.1) can be modified for the centered sample covariance matrices

$$\begin{aligned}
& a_{np}^{-2}\Big\|\big(\mathbf{X}_n\mathbf{X}_n' - E\mathbf{X}_n\mathbf{X}_n')\big) - \Big(\mathbf{X}_n^{(m)}(\mathbf{X}_n^{(m)})' - E\mathbf{X}_n^{(m)}(\mathbf{X}_n^{(m)})'\Big)\Big\|_2 \\
\le\ & a_{np}^{-2} \max_{i=1,\dots,p} \sum_{j=1}^p \Big| \sum_{l=0}^\infty \sum_{k\vee(k+j-i)>m} h_{kl}h_{k+j-i,l} \sum_{t=1}^n (Z_{i-k,t-l}^2 - EZ^2)\Big| + I_n^{(2)} + I_n^{(3)} \\
=\ & I_n^{(1)} + I_n^{(2)} + I_n^{(3)}\,.
\end{aligned}$$

Here $I_n^{(i)}$, $i = 2,3$, are the same as in (5.1) and we also recycle the notation $I_n^{(1)}$ in the mean corrected case.

*Bounds for $P(I_n^{(1)} > \varepsilon)$.* In this case, again recycling notation,

$$\begin{aligned}
I_n^{(1)} &\le c a_{np}^{-2} \max_{i=1,\dots,p} \sum_{l=0}^\infty \sum_{k=m+1}^\infty |h_{kl}| \Big| \sum_{t=1}^n (Z_{i-k,t-l}^2 - EZ^2)\Big| \\
&\quad + c a_{np}^{-2} \max_{i=1,\dots,p} \sum_{l=0}^\infty \sum_{s=m+1}^\infty \sum_{k=1}^m |h_{sl}h_{kl}| \Big| \sum_{t=1}^n (Z_{i-k,t-l}^2 - EZ^2)\Big| \\
&= I_n^{(11)} + I_n^{(12)}\,.
\end{aligned}$$

We will again focus on $I_n^{(11)}$, the case $I_n^{(12)}$ can be treated analogously. Applying Hölder's inequality of order $\gamma > \alpha/2$, we have for $i \le p$,

$$
\begin{aligned}
&\Big(\sum_{l=0}^{\infty}\sum_{k=m+1}^{\infty}|h_{kl}|\Big|\sum_{t=1}^{n}(Z^2_{i-k,t-l}-EZ^2)\Big|\Big)^{\gamma}\\
&\le\ c\sum_{l=0}^{\infty}\sum_{k=m+1}^{\infty}|h_{kl}|\Big|\sum_{t=1}^{n}(Z^2_{i-k,t-l}-EZ^2)\Big|^{\gamma}\Big(\sum_{l=0}^{\infty}\sum_{k=m+1}^{\infty}|h_{kl}|\Big)^{\gamma-1}\\
&\le\ c\sum_{l=0}^{\infty}\sum_{k=m+1}^{\infty}|h_{kl}|\Big|\sum_{t=1}^{n}(Z^2_{i-k,t-l}-EZ^2)\Big|^{\gamma}.
\end{aligned}
$$

Therefore we have for $\varepsilon > 0$,

$$
\begin{aligned}
&P(|I_n^{(11)}|^{\gamma}>\varepsilon)\\
&\le\ P\Big(c\max_{i=1,\dots,p}\sum_{l=0}^{\infty}\sum_{k=m+1}^{\infty}|h_{kl}|\Big|\sum_{t=1}^{n}(Z^2_{i-k,t-l}-EZ^2)\Big|^{\gamma}>\varepsilon a_{np}^{2\gamma}\Big)\\
&\le\ \sum_{i=1}^{p}P\Big(ca_{np}^{-2\gamma}\sum_{l=0}^{\infty}\sum_{k=m+1}^{\infty}|h_{kl}|\Big|\sum_{t=1}^{n}(Z^2_{i-k,t-l}-EZ^2)\Big|^{\gamma}I_{\{||h_{kl}|\sum_{s=1}^{n}(Z^2_{i-k,s-l}-EZ^2)|\le a_{np}^2\}}>\varepsilon\Big)\\
&\quad +p\sum_{l=0}^{\infty}\sum_{k=m+1}^{\infty}P\Big(|h_{kl}|\sum_{t=1}^{n}(Z_t^2-EZ^2)\Big|>a_{np}^2\Big)\\
&=\ P_n^{(1)}+P_n^{(2)}\,.
\end{aligned}
$$

An application of the uniform large deviation results of Theorem A.1 yields

$$
P_n^{(2)}\le c\sum_{l=0}^{\infty}\sum_{k=m+1}^{\infty}|h_{kl}|^{\alpha/2}[npP(Z^2>a_{np}^2)]\,.
$$

The right-hand side is uniformly bounded for $n$ and converges to zero as $m\to\infty$. Next we apply Markov's inequality to $P_n^{(1)}$:

$$
P_n^{(1)}\ \le\ cp\sum_{l=0}^{\infty}\sum_{k=m+1}^{\infty}|h_{kl}|^{1-\gamma}\ E\Big|a_{np}^{-2}|h_{kl}|\sum_{t=1}^{n}(Z_t^2-EZ^2)\Big|^{\gamma}I_{\{|h_{kl}||\sum_{s=1}^{n}(Z_s^2-EZ^2)|\le a_{np}^2\}}\,.
$$

Choose $d_n = a_n^2 s_n$ for $s_n\to\infty$ arbitrarily slowly. An application of Lemma A.2 yields, assuming that $h_{kl}\neq 0$,

$$
\begin{aligned}
&p\,E\Big|a_{np}^{-2}|h_{kl}|\sum_{t=1}^{n}(Z_t^2-EZ^2)\Big|^{\gamma}I_{\{d_n\le|\sum_{t=1}^{n}(Z_t^2-EZ^2)|\le|h_{kl}|^{-1}a_{np}^2\}}\\
&\le\ |h_{kl}|^{\alpha/2}\,[npP(Z^2>a_{np}^2)]\le c\,|h_{kl}|^{\alpha/2}\,.
\end{aligned}
$$

On the other hand, choosing $\gamma$ close to $\alpha/2$ and a small $\xi>0$ such that $\gamma(1-\xi)<\alpha/2$.

$$
\begin{aligned}
&p\,E\Big|a_{np}^{-2}|h_{kl}|\sum_{t=1}^{n}(Z_t^2-EZ^2)\Big|^{\gamma}I_{\{|\sum_{t=1}^{n}(Z_t^2-EZ^2)|\le d_n\}}\\
&\le\ p\,d_n^{\gamma\xi}a_n^{2\gamma(1-\xi)}(|a_{np}^{-2}|h_{kl}|)^{\gamma}E\Big|a_n^{-2}\sum_{t=1}^{n}(Z_t^2-EZ^2)\Big|^{\gamma(1-\xi)}\\
&\le\ cp\,s_n^{\gamma\xi}(a_n/a_{np})^{2\gamma}|h_{kl}|^{\gamma}\\
&\le\ p^{1-2\gamma/\alpha+\delta}s_n^{\gamma\xi}|h_{kl}|^{\gamma}\,.
\end{aligned}
$$

Here we used the uniform integrability of $(a_n^{-2}\sum_{t=1}^n(Z_t^2-EZ^2))$ of order $\gamma(1-\xi)<\alpha/2$ and the Potter bounds (see Bingham et al. [4]) for any small $\delta>0$. Combining the above bounds, we have

$$P_n^{(1)}\le c\sum_{l=0}^{\infty}\sum_{k=m+1}^{\infty}\Big[|h_{kl}|^{1-\gamma+\alpha/2}+|h_{kl}|p^{1-2\gamma/\alpha+\delta}s_n^{\gamma\xi}\Big]\,.$$

If we choose $\gamma$ close to $\alpha/2$, $\delta$ sufficiently small and $s_n\to\infty$ slowly, the right-hand side converges to zero by first letting $n\to\infty$ and then $m\to\infty$.

*Bounds for* $P(I_n^{(2)}>\varepsilon)$. We focus on this case; the proof for $I_n^{(3)}$ is again analogous. One can follow the lines of the proof in the case $\alpha\in(1,2)$, choosing $\gamma\in(\alpha,4)$ close to $\alpha$. The same argument goes through for $P_n^{(42)}$, using (2.13). For $P_n^{(41)}$, one needs to use the moment inequality (A.4) from Lemma A.4, Karamata's theorem and (2.13). Then one gets

$$\begin{aligned}P_n^{(41)} &\le c\,p^{\gamma}n^{\gamma/2}a_{np}^{-2\gamma}\sum_S|h_{k_1,l_1}h_{k_2,l_2}|E|Z_1Z_2|^{\gamma}I_{\{|h_{k_1,l_1}h_{k_2,l_2}|\,|Z_1Z_2|\le a_{np}^2\}}\\ &\le c\,n^{\gamma/2}p^{\gamma}\sum_S|h_{k_1,l_1}h_{k_2,l_2}|^{1-\gamma+\alpha}P(|Z_1Z_2|>a_{np}^2)\\ &\le c\,n^{\gamma/2}\,p^{\gamma+1}P(|Z_1Z_2|>a_{np}^2)\to0\,,\quad n\to\infty\,.\end{aligned}$$

This concludes the proof in the case $\alpha\in(2,4)$. □

**Step 2: Approximation of $\mathbf{X}_n^{(m)}(\mathbf{X}_n^{(m)})'$ by an iid sum with deterministic matrix weights.** Let $M^{(m)}$ be the upper left $(m+1)\times(m+1)$ corner matrix of $M$ defined in (2.1), i.e.

$$M^{(m)}=(M_{ij})_{i,j\le m}\,.$$

Denote the ordered eigenvalues of $M^{(m)}$ by

$$v_1^{(m)}\ge\cdots\ge v_{m+1}^{(m)}\,,$$

and let $r_m$ be the rank of $M^{(m)}$. Since the trace of $M$ is finite as a compact operator on $l^2$ a standard argument from matrix operator theory shows for each $i$, $v_i^{(m)}\to v_i$ as $m\to\infty$; see Gohberg et al. [10].

Next define the $p\times p$-matrices $(M_s^{(m)})$, $s\in\mathbb{Z}$, via

$$M_{s,ij}^{(m)}=\begin{cases}M_{i-s,j-s}\,, & i,j=s,\ldots,m+s\,,\\ 0\,, & \text{otherwise}\end{cases}\,. \tag{5.4}$$

For $s\ge1$, $M_s^{(m)}$ has rank $r_m$ and has the same eigenvalues $v_1^{(m)},\ldots,v_{m+1}^{(m)}$ as $M^{(m)}$.

**Lemma 5.2.** *Assume the conditions of Theorem 3.1.*

(1) *For $\alpha\in(0,2)$, we have*

$$a_{np}^{-2}\Big\|\mathbf{X}_n^{(m)}(\mathbf{X}_n^{(m)})'-\sum_{i=1}^pD_iM_i^{(m)}\Big\|_2\xrightarrow{P}0\,,\quad n\to\infty\,.$$

(2) *For $\alpha\in(2,4)$, we have*

$$a_{np}^{-2}\Big\|\big(\mathbf{X}_n^{(m)}(\mathbf{X}_n^{(m)})'-E\mathbf{X}_n^{(m)}(\mathbf{X}_n^{(m)})'\big)-\sum_{i=1}^p(D_i-ED_i)M_i^{(m)}\Big\|_2\xrightarrow{P}0\,,\quad n\to\infty\,.$$

*Proof.* **(1) The case $\alpha \in (0,2)$.** The $(i,j)$th element of $\mathbf{X}_n^{(m)}(\mathbf{X}_n^{(m)})'$ can be decomposed as follows:

$$\begin{aligned}
\sum_{t=1}^n X_{it}^{(m)} X_{jt}^{(m)} &= \sum_{t=1}^n \sum_{k_1=0}^m \sum_{l_1=0}^\infty \sum_{k_2=0}^m \sum_{l_2=0}^\infty h_{k_1,l_1} h_{k_2,l_2}\, Z_{i-k_1,t-l_1} Z_{j-k_2,t-l_2} \\
&= \sum_{t=1}^n \sum_{k=0}^m \sum_{l=0}^\infty h_{k,l} h_{j-i+k,l}\, Z^2_{i-k,t-l} \\
&\quad + \sum_{t=1}^n \sum_{0\le k_1,k_2\le m} \sum_{0\le l_1\ne l_2<\infty} h_{k_1,l_1} h_{k_2,l_2} Z_{i-k_1,t-l_1} Z_{j-k_2,t-l_2} \\
&\quad + \sum_{t=1}^n \sum_{i-k_1\ne j-k_2} \sum_{l=0}^\infty h_{k_1,l} h_{k_2,l} Z_{i-k_1,t-l} Z_{j-k_2,t-l} \\
&= I_{ij}^{(1)} + I_{ij}^{(2)} + I_{ij}^{(3)}\,.
\end{aligned}$$

Thus $\mathbf{X}_n^{(m)}(\mathbf{X}_n^{(m)})'$ can be decomposed into the sum of 3 quadratic matrices:

$$\mathbf{X}_n^{(m)}(\mathbf{X}_n^{(m)})' = I_n^{(1)} + I_n^{(2)} + I_n^{(3)}\,.$$

We start by proving that

$$a_{np}^{-2}\Delta_n = a_{np}^{-2}\big\|I_n^{(1)} - \sum_{i=1}^p D_i M_i^{(m)}\big\|_2 \xrightarrow{P} 0\,. \tag{5.5}$$

We observe that

$$\begin{aligned}
a_{np}^{-2}\Delta_n &\le a_{np}^{-2}\big\|I_n^{(1)} - \sum_{i=-m}^p D_i M_i^{(m)}\big\|_2 + a_{np}^{-2}\big\|\sum_{i=-m}^0 D_i M_i^{(m)}\big\|_2 \\
&= a_{np}^{-2}\Delta_{n1} + a_{np}^{-2}\Delta_{n2}\,.
\end{aligned} \tag{5.6}$$

We have

$$a_{np}^{-2}\Delta_{n2} \le a_{np}^{-2} \max_{i=-m,\dots,0} D_i \sum_{i=-m}^0 \|M_i^{(m)}\|_2 \xrightarrow{P} 0\,. \tag{5.7}$$

This follows because $\|M_i^{(m)}\|_2 < \infty$ for each $i$ and since $a_n^{-2} D_i \xrightarrow{d} \xi_{\alpha/2}$ for an $\alpha/2$-stable random variable $\xi_{\alpha/2}$ as $n\to\infty$, hence $a_{np}^{-2} D_i \xrightarrow{P} 0$ by virtue of $p_n\to\infty$.

Observe that the $(i,j)$th element of $I_n^{(1)} - \sum_{i=-m}^p D_i M_i^{(m)}$ is

$$\begin{aligned}
\sum_{k=0}^m \sum_{l=0}^\infty h_{kl} h_{j-i+k,l}\Big(\sum_{t=1}^n Z^2_{i-k,t-l} - D_{i-k}\Big) &= \sum_{k=0}^m \sum_{l=0}^\infty h_{kl} h_{j-i+k,l}\Big(\sum_{t=1}^l Z^2_{i-k,t-l} - \sum_{t=n-l+1}^n Z^2_{i-k,t}\Big) \\
&= I_{ij}^{(11)} - I_{ij}^{(12)}\,.
\end{aligned}$$

For $a_{np}^{-2}\Delta_{n1} \xrightarrow{P} 0$ it suffices to show that

$$a_{np}^{-2}\|I_n^{(1i)}\|_2 \xrightarrow{P} 0\,, \quad i=1,2.$$

We will show the limit relation for $i=1$; the case $i=2$ is analogous. Interpreting $h_{kl}=0$ for $k>m$ and $k<0$ in the sequel, we observe that

$$\begin{aligned}
a_{np}^{-2}\|I_n^{(11)}\|_2 &\le a_{np}^{-2}\max_{i=1,\ldots,p}\sum_{j=1}^{p}\sum_{k=0}^{m}\sum_{l=0}^{\infty}|h_{kl}h_{j-i+k,l}|\sum_{t=1}^{l}Z_{i-k,t-l}^2\\
&\le a_{np}^{-2}\max_{i=1,\ldots,p}\sum_{k'=0}^{m}\sum_{k=0}^{m}\sum_{l=0}^{\infty}|h_{kl}h_{k',l}|\sum_{t=1}^{l}Z_{i-k,t-l}^2\\
&\le c\,a_{np}^{-2}\max_{i=1,\ldots,p}\sum_{l=0}^{\infty}\sum_{k=0}^{m}|h_{kl}|\sum_{t=1}^{l}Z_{i-k,t-l}^2\,. \qquad (5.8)
\end{aligned}$$

For $\varepsilon>0$ and a sequence $(Z_t)$ of iid copies of $Z$, we have

$$\begin{aligned}
P\Big(a_{np}^{-2}\|I_n^{(11)}\|_2>\varepsilon\Big) &\le \sum_{i=1}^{p}P\Big(c\sum_{l=0}^{\infty}\sum_{k=0}^{m}|h_{kl}|\sum_{t=1}^{l}Z_{i-k,t-l}^2>\varepsilon a_{np}^2\Big)\\
&\le p\sum_{k=0}^{m}P\Big(c\sum_{l=0}^{\infty}|h_{kl}|\sum_{t=1}^{l}Z_t^2>\varepsilon a_{np}^2/m\Big)\,. \qquad (5.9)
\end{aligned}$$

We focus on the summand with $k=0$; the other cases are similar. We have for any $c>0$, by a standard result (see Mikosch and Samorodnitsky [12]),

$$\begin{aligned}
p\,P\Big(c\sum_{l=0}^{\infty}|h_{0l}|\sum_{t=0}^{l}Z_t^2>a_{np}^2\Big) &= p\,P\Big(c\sum_{t=0}^{\infty}Z_t^2\sum_{l=t}^{\infty}|h_{0l}|>a_{np}^2\Big)\\
&\sim c\,p\,P(Z^2>a_{np}^2)\sum_{t=0}^{\infty}\Big(\sum_{l=t}^{\infty}|h_{0l}|\Big)^{\alpha/2}\\
&\le c\,n^{-1}\sum_{t=0}^{\infty}\Big(\sum_{l=t}^{\infty}|h_{0l}|\Big)^{\alpha/2}\to 0\,,\quad n\to\infty\,,
\end{aligned}$$

where in the last step we used condition (2.8). This proves relation (5.5).

It remains to show that

$$a_{np}^{-2}\|I_n^{(i)}\|_2\stackrel{P}{\to}0\,,\quad n\to\infty\,,\quad i=2,3\,.$$

However, the proof is almost identical to the proof of the corresponding result for $I_n^{(i)}$, $i=2,3$, in Lemma 5.1. To indicate this fact consider the case $i=2$. Note that for $\varepsilon>0$,

$$P(\|I_n^{(2)}\|_2>\varepsilon)\le p\sum_{i=1}^{p}P(U_i>\varepsilon a_{np}^2)$$

where

$$
\begin{aligned}
U_i &= \sum_{j=1}^{p} \sum_{0\le l_1\neq l_2<\infty} \sum_{0\le k_1,k_2\le m} |h_{k_1,l_1} h_{k_2,l_2}| \Big| \sum_{t=1}^{n} Z_{i-k_1,t-l_1} Z_{j-k_2,t-l_2} \Big| \\
&\le \sum_{j=1}^{p} \sum_{0\le l_1\neq l_2<\infty} \sum_{0\le k_1,k_2\le m} |h_{k_1,l_1} h_{k_2,l_2}| \\
&\quad \Big( \Big| \sum_{t=1}^{n} Z_{i-k_1,t-l_1} Z_{j-k_2,t-l_2} I_{\{|h_{k_1,l_1}h_{k_2,l_2}|\,|Z_{i-k_1,t-l_1}Z_{j-k_2,t-l_2}|>a_{np}^2\}} \Big| \\
&\quad + \Big| \sum_{t=1}^{n} Z_{i-k_1,t-l_1} Z_{j-k_2,t-l_2} I_{\{|h_{k_1,l_1}h_{k_2,l_2}|\,|Z_{i-k_1,t-l_1}Z_{j-k_2,t-l_2}|\le a_{np}^2\}} \Big| \Big) \,.
\end{aligned}
$$

Now we can follows the lines of the proof of Lemma 5.1; note that the corresponding results for $I_n^{(i)}$, $i=1,2$, were proved for fixed $m$ and as $n\to\infty$.

The proof of the lemma is now complete for $\alpha\in(0,2)$.

**(2) The case $\alpha\in(2,4)$.** Using the fact that $EZ_1Z_2=0$, the $(i,j)$th element of $\mathbf{X}_n^{(m)}(\mathbf{X}_n^{(m)})' - E\mathbf{X}_n^{(m)}(\mathbf{X}_n^{(m)})'$ can be decomposed as follows:

$$
\begin{aligned}
\sum_{t=1}^{n} (X_{it}^{(m)} X_{jt}^{(m)} - E X_{it}^{(m)} X_{jt}^{(m)}) &= \sum_{t=1}^{n}\sum_{k=0}^{m}\sum_{l=0}^{\infty} h_{k,l} h_{j-i+k,l}\,(Z_{i-k,t-l}^2 - EZ^2) + I_{ij}^{(2)} + I_{ij}^{(3)} \\
&= \widetilde{I}_{ij}^{(1)} + I_{ij}^{(2)} + I_{ij}^{(3)}\,,
\end{aligned}
$$

Thus $\mathbf{X}_n^{(m)}(\mathbf{X}_n^{(m)})' - E\mathbf{X}_n^{(m)}(\mathbf{X}_n^{(m)})'$ can be decomposed into the sum of 3 quadratic matrices:

$$
\mathbf{X}_n^{(m)}(\mathbf{X}_n^{(m)})' = \widetilde{I}_n^{(1)} + I_n^{(2)} + I_n^{(3)}\,,
$$

where $I_n^{(2)}, I_n^{(3)}$ are the same as in the proof of part (1).

One can show for $\varepsilon>0$ that $P(\|I_n^{(i)}\|_2>\varepsilon)\to 0$, $i=1,2$, as $n\to\infty$, following the lines of the corresponding results in the proof of Lemma 5.1. It remains to show that

$$
a_{np}^{-2}\widetilde{\Delta}_n = a_{np}^{-2} \Big\| \widetilde{I}_n^{(1)} - \sum_{i=1}^{p} (D_i - ED) M_i^{(m)} \Big\|_2 \xrightarrow{P} 0\,. \tag{5.10}
$$

We may essentially follow the proof of (5.5), taking into account the centering of the quantities involved. To start with,

$$
a_{np}^{-2}\widetilde{\Delta}_n \;=\; a_{np}^{-2} \big\| \widetilde{I}_n^{(1)} - \sum_{i=-m}^{p} (D_i - ED) M_i^{(m)} \big\|_2 + o_p(1)\,, \quad n\to\infty\,,
$$

by an argument similar to (5.6) and (5.7), observing that $a_n^{-2}(D_1-ED)\xrightarrow{d}\xi_{\alpha/2}$ for an $\alpha/2$-stable random variable which is totally skewed to the right, hence $a_{np}^{-2}(D_1-ED)\xrightarrow{P}0$ as $n\to\infty$.

Next, observe that

$$
\widetilde{I}_n^{(1)} - \sum_{i=-m}^{p} (D_i - ED) M_i^{(m)} = \widetilde{I}_n^{(11)} - \widetilde{I}_n^{(12)}\,,
$$

where the $(i,j)$th element of the right-hand side is given by

$$\begin{aligned}
&\sum_{k=0}^{m}\sum_{l=0}^{\infty} h_{kl}h_{j-i+k,l}\Big(\sum_{t=1}^{n}(Z^2_{i-k,t-l}-EZ^2)-(D_{i-k}-ED)\Big)\\
&\quad= \sum_{k=0}^{m}\sum_{l=0}^{\infty} h_{kl}h_{j-i+k,l}\Big(\sum_{t=1}^{l}Z^2_{i-k,t-l}-EZ^2)-\sum_{t=n-l+1}^{n}(Z^2_{i-k,t}-EZ^2)\Big)\\
&\quad= \widetilde{I}_{ij}^{(11)}-\widetilde{I}_{ij}^{(12)}\,.
\end{aligned}$$

Thus it suffices to show that

$$a_{np}^{-2}\|\widetilde{I}_n^{(1i)}\| \xrightarrow{P} 0, \quad i=1,2, \quad n\to\infty\,. \tag{5.11}$$

Again, we will focus on the case $i=1$. Similar arguments as for (5.8) and (5.9) yield

$$\begin{aligned}
a_{np}^{-2}\|I_n^{(11)}\|_2 &\le a_{np}^{-2}\max_{i=1,\ldots,p}\sum_{j=1}^{p}\sum_{k=0}^{m}\sum_{l=0}^{\infty}|h_{kl}h_{j-i+k,l}|\Big|\sum_{t=1}^{l}(Z^2_{i-k,t-l}-EZ^2)\Big|\\
&\le c\,a_{np}^{-2}\max_{i=1,\ldots,p}\sum_{l=0}^{\infty}\sum_{k=0}^{m}|h_{kl}|\Big|\sum_{t=1}^{l}(Z^2_{i-k,t-l}-EZ^2)\Big|\,,
\end{aligned}$$

and for $\varepsilon>0$,

$$\begin{aligned}
P\Big(a_{np}^{-2}\|I_n^{(11)}\|_2>\varepsilon\Big) &\le p\sum_{k=0}^{m}P\Big(c\sum_{l=0}^{\infty}|h_{kl}|\Big|\sum_{t=0}^{l}(Z_t^2-EZ^2)\Big|>\varepsilon a_{np}^2/m\Big)\\
&\le p\sum_{k=0}^{m}P\Big(c\sum_{l=0}^{\infty}|h_{kl}|\sum_{t=0}^{l}|Z_t^2-EZ^2|>\varepsilon a_{np}^2/m\Big)\,.
\end{aligned}$$

We focus on the summand with $k=0$; the other cases are analogous. We have for any $c>0$, by a standard result (see Mikosch and Samorodnitsky [12]),

$$\begin{aligned}
p\,P\Big(c\sum_{l=0}^{\infty}|h_{0l}|\sum_{t=0}^{l}|Z_t^2-EZ^2|>a_{np}^2\Big) &= p\,P\Big(c\sum_{t=0}^{\infty}|Z_t^2-EZ^2|\sum_{l=t}^{\infty}|h_{0l}|>a_{np}^2\Big)\\
&\sim c\,p\,P(Z^2>a_{np}^2)\sum_{t=0}^{\infty}\Big(\sum_{l=t}^{\infty}|h_{0l}|\Big)^{\alpha/2}\\
&\le c\,n^{-1}\sum_{t=0}^{\infty}\Big(\sum_{l=t}^{\infty}|h_{0l}|\Big)^{\alpha/2}\to 0\,,\quad n\to\infty\,.
\end{aligned}$$

In the last step we used condition (2.8). Thus we proved (5.11). This concludes the proof of the lemma for $\alpha\in(2,4)$. □

**Step 3: Approximation of $\sum_{i=1}^{p}D_iM_i^{(m)}$ by a block diagonal matrix.**

**The case $\alpha\in(0,2)$.** From (2.4) recall the definition of the order statistics

$$D_{(p)}=D_{L_p}<\cdots<D_{(1)}=D_{L_1}\quad \text{a.s.}$$

of the iid sequence $D_1,\ldots,D_p$ defined in (1.4). Here we assume without loss of generality that there are no ties in the sample. Otherwise, if two or more of the $D_i$'s are equal, randomize the corresponding $L_i$'s over the respective indices.

We choose an integer sequence $k=k_p\to\infty$ such that $k_p^2=o(p)$ as $n\to\infty$ and define the event

$$A_n=\{|L_i-L_j|>m+1\,,i,j=1,\ldots,k\,,i\neq j\}\,. \tag{5.12}$$

Since the $D_i$'s are iid, $L_1, \ldots, L_k$ have a uniform distribution on the set of distinct $k$-tuples from $(1, \ldots, p)$ and

$$P(A_n^c) \le k(k-1)\frac{pm(p-2)\ldots(p-k+1)}{p(p-1)\ldots(p-k+1)} \le \frac{k^2 m}{p-1} \to 0\,, \quad n \to \infty\,. \tag{5.13}$$

On the event $A_n$, the matrix $\sum_{i=1}^k D_{L_i} M_{L_i}^{(m)}$ is block diagonal and has positive eigenvalues $D_{(i)} v_j^{(m)}$, $i = 1, \ldots, k$, $j = 1, \ldots, r_m$.

In the next step of the proof we approximate $\sum_{i=1}^p D_i M_i^{(m)}$ by the matrix $\sum_{i=1}^k D_{L_i} M_{L_i}^{(m)}$ which is block diagonal with high probability.

**Lemma 5.3.** *Assume $\alpha \in (0,2)$ and that the conditions of Theorem 3.1 are satisfied. Consider an integer sequence $(k_p)$ such that $k_p \to \infty$ and $k_p^2 = o(p)$ as $n \to \infty$. Then*

$$a_{np}^{-2}\Big\| \sum_{i=1}^p D_i M_i^{(m)} - \sum_{i=1}^k D_{L_i} M_{L_i}^{(m)} \Big\|_2 \stackrel{P}{\to} 0\,, \quad n \to \infty\,.$$

*Proof.* We have

$$a_{np}^{-2}\Big( \sum_{i=1}^p D_i M_i^{(m)} - \sum_{i=1}^k D_{L_i} M_{L_i}^{(m)} \Big) = a_{np}^{-2} \sum_{i=k+1}^p D_{L_i} M_{L_i}^{(m)}\,,$$

and therefore it suffices to show that the right-hand side converges to zero in probability. Then for $\delta > 0$,

$$P\Big( a_{np}^{-2}\Big\| \sum_{i=k+1}^p D_{L_i} M_{L_i}^{(m)} \Big\|_2 > \delta \Big) \quad \le \quad P\Big( c\, a_{np}^{-2} \sum_{i=k+1}^p D_{(i)} > \delta \Big)\,. \tag{5.14}$$

We will show that the right-hand side converges to zero as $n \to \infty$.

We conclude from Lemma A.3 that

$$\sum_{i=1}^p \varepsilon_{a_{np}^{-2} D_i} \stackrel{d}{\to} \sum_{i=1}^\infty \varepsilon_{\Gamma_i^{-2/\alpha}}\,, \quad n \to \infty\,,$$

where $(\Gamma_i)$ is an increasing enumeration of the points of a homogeneous Poisson process on $(0, \infty)$. A continuous mapping argument (Resnick [16], Theorem 7.1) shows that for every $\gamma > 0$,

$$a_{np}^{-2}\Big( \sum_{i=1}^p D_i I_{\{a_{np}^{-2} D_i > \gamma\}}\,, \sum_{i=1}^p D_i \Big) \stackrel{d}{\to} \Big( \sum_{i=1}^\infty \Gamma_i^{-2/\alpha} I_{\{\Gamma_i^{-2/\alpha} > \gamma\}}\,, \sum_{i=1}^\infty \Gamma_i^{-2/\alpha} \Big)$$

provided

$$\lim_{\xi \downarrow 0} \limsup_{n \to \infty} p\, a_{np}^{-2}\, E D I_{\{a_{np}^{-2} D \le \xi\}} = 0, \tag{5.15}$$

and hence

$$a_{np}^{-2} \sum_{i=1}^p D_i I_{\{a_{np}^{-2} D_i \le \gamma\}} \stackrel{d}{\to} \sum_{i=1}^\infty \Gamma_i^{-2/\alpha} I_{\{\Gamma_i^{-2/\alpha} \le \gamma\}}\,.$$

But we also have

$$a_{np}^{-2} \sum_{i=k+1}^p D_{(i)} = a_{np}^{-2} \sum_{i=1}^p D_i I_{\{a_{np}^{-2} D_i < a_{np}^{-2} D_{(k)}\}}\,,$$

and $a_{np}^{-2}D_{(k)} \stackrel{P}{\to} 0$ as $n\to\infty$. For $\gamma>0$, we write $B_n=\{a_{np}^{-2}D_{(k)}\le\gamma\}$. Then $P(B_n^c)\to 0$ as $n\to\infty$ and

$$\begin{aligned} P\Big(ca_{np}^{-2}\sum_{i=k+1}^{p} D_{(i)}>\delta\Big) &\le P\Big(B_n\cap\{c\,a_{np}^{-2}\sum_{i=1}^{p} D_i I_{\{a_{np}^{-2}D_i\le\gamma\}}>\delta\}\Big)+o(1)\\ &\le P\Big(c\,a_{np}^{-2}\sum_{i=1}^{p} D_i I_{\{a_{np}^{-2}D_i\le\gamma\}}>\delta\Big)+o(1)\\ &\to P\Big(c\sum_{i=1}^{\infty}\Gamma_i^{-2/\alpha} I_{\{\Gamma_i^{-2/\alpha}\le\gamma\}}>\delta\Big)\,,\quad n\to\infty\,,\\ &\to 0\,,\quad \gamma\downarrow 0\,. \end{aligned}$$

Therefore the right-hand side in (5.14) converges to zero if (5.15) holds.

Thus it remains to show (5.15). By Karamata's theorem, as $n\to\infty$,

$$p\,a_{np}^{-2}\,EDI_{\{D\le\xi a_{np}^2\}} \;\le\; np\,a_{np}^{-2}\,EZ^2 I_{\{Z^2\le a_{np}^2\xi\}}\sim\xi\,np\,P(Z^2>a_{np}\xi)\sim\xi^{1-\alpha/2}\,,$$

and the right-hand side converges to zero as $\xi\downarrow 0$. Then (5.15) follows and the proof of the lemma is finished. □

**The case** $\alpha\in(2,4)$**.** Recall that the order statistics of $\widetilde{D}_i=|D_i-ED|$, $i=1,\ldots,p$, are denoted by

$$\widetilde{D}_{(p)}=\widetilde{D}_{\ell_p}\le\cdots\le\widetilde{D}_{(1)}=\widetilde{D}_{\ell_1}\quad\text{a.s.,}\tag{5.16}$$

where we again assume without loss of generality that there are no ties in the sample.

We choose an integer sequence $k=k_p\to\infty$ such that $k_p^2=o(p)$ as $n\to\infty$ and define the event

$$\widetilde{A}_n=\{|\ell_i-\ell_j|>m+1\,,i,j=1,\ldots,k\,,i\ne j\}\,.\tag{5.17}$$

As for $A_n$, we have

$$P(\widetilde{A}_n^c)\le\frac{k^2m}{p-1}\to 0\quad,\quad n\to\infty\,.\tag{5.18}$$

On the event $\widetilde{A}_n$, the matrix $\sum_{=1}^{k}(D_{\ell_i}-ED)M_{\ell_i}^{(m)}$ is block diagonal and has the non-zero eigenvalues $(D_{\ell_i}-ED)v_j^{(m)}$, $i=1,\ldots,k$, $j=1,\ldots,r_m$.

**Lemma 5.4.** *Assume $\alpha\in(2,4)$ and that the conditions of Theorem 3.1 are satisfied. Consider an integer sequence $(k_p)$ such that $k_p\to\infty$ and $k_p^2=o(p)$ as $n\to\infty$. Then*

$$a_{np}^{-2}\Big\|\sum_{i=1}^{p}(D_i-ED)M_i^{(m)}-\sum_{i=1}^{k}(D_{\ell_i}-ED)M_{\ell_i}^{(m)}\Big\|_2\stackrel{P}{\to}0\,,\quad n\to\infty\,.$$

*Proof.* As a first step in the proof we show the following relation for every $\delta>0$,

$$\lim_{\gamma\downarrow 0}\limsup_{n\to\infty}P\Big(a_{np}^{-2}\Big\|\sum_{i=1}^{p}(D_i-ED)I_{\{a_{np}^{-2}|D_i-ED|\le\gamma\}}M_i^{(m)}\Big\|_2>\delta\Big)=0\,.\tag{5.19}$$

We observe that we can divide the index set $i=1,\ldots,p$ of the involved sums into disjoint subsets $I_1=\{1,m+2,2m+3,\ldots\}\cap\{1,\ldots,p\}$, $I_2=\{2,m+3,2m+4,\ldots\}\cap\{1,\ldots,p\}$, etc. For $\gamma>0$, due to the construction of the matrices $M_i^{(m)}$ (see (5.4)), the sums

$$T_j=a_{np}^{-2}\sum_{i\in I_j}(D_i-ED)I_{\{a_{np}^{-2}|D_i-ED|\le\gamma\}}M_i^{(m)}\,,\quad j=1,\ldots,m+1\,,$$

constitute block diagonal matrices with norm

$$\|T_j\|_2 = a_{np}^{-2} \max_{i\in I_j} |D_i - ED| I_{\{a_{np}^{-2}|D_i-ED|\le\gamma\}}\, v_1\,.$$

An application of Markov's inequality yields

$$\begin{aligned}
&P\Big(a_{np}^{-2}\Big\|\sum_{i=1}^p (D_i-ED) I_{\{a_{np}^{-2}|D_i-ED|\le\gamma\}} M_i^{(m)}\Big\|_2 > \delta\Big)\\
&\le \sum_{j=1}^{m+1} P\Big(\|T_j\|_2 > \delta/(m+1)\Big)\\
&\le c\, a_{np}^{-4}\sum_{j=1}^{m+1} E\max_{i\in I_j} |D_i-ED|^2 I_{\{a_{np}^{-2}|D_i-ED|\le\gamma\}}\\
&\le c\, a_{np}^{-4}\sum_{i=1}^{p} E|D_i-ED|^2 I_{\{a_{np}^{-2}|D_i-ED|\le\gamma\}}\\
&= c\, p a_{np}^{-4} E|D-ED|^2 I_{\{a_{np}^{-2}|D-ED|\le\gamma\}}\,. \qquad (5.20)
\end{aligned}$$

An application of (A.2) yields for $d_n = a_n^2 s_n$, any sequence $(s_n)$ such that $s_n\to\infty$,

$$p\, a_{np}^{-4} E|D-ED|^2 I_{\{d_n\le |D-ED|\le a_{np}^2\gamma\}} \le cnp\,\gamma^2 P(Z^2 > a_{np}^2\gamma) \sim c\gamma^{(4-\alpha)/2}\,,\quad n\to\infty\,,$$

and the right-hand side converges to zero as $\gamma\to 0$. On the other hand, for $\epsilon>0$ arbitrarily small and $s_n\to\infty$ sufficiently slowly,

$$p\, a_{np}^{-4} E|D-ED|^2 I_{\{\le |D-ED|\le d_n\}} \le p s_n^2 (a_n/a_{np})^4 \le c\, s_n^2 p^{1-4/\alpha+\epsilon}\to 0\,,\quad n\to\infty\,.$$

Thus we proved that (5.20) converges to zero by first letting $n\to\infty$ and then $\gamma\to 0$. This proves (5.19).

In view of (5.19), the lemma is proved if we can show that for every $\delta>0$ and $\gamma\in(0,\gamma_0(\delta))$ for a sufficiently small $\gamma_0(\delta)$,

$$\lim_{n\to\infty} P\Big(a_{np}^{-2}\Big\|\sum_{i=1}^p (D_i-ED)M_i^{(m)} I_{\{|D_i-ED|>\gamma a_{np}^2\}} - \sum_{i=1}^k (D_{\ell_i}-ED) M_{\ell_i}^{(m)}\Big\|_2 > \delta\Big) = 0\,. \qquad (5.21)$$

If

$$\begin{aligned}
0 &\ne \sum_{i=1}^p (D_i-ED)M_i^{(m)} I_{\{|D_i-ED|>\gamma a_{np}^2\}} - \sum_{i=1}^k (D_{\ell_i}-ED) I_{\{|D_{\ell_i}-ED|>\gamma a_{np}^2\}} M_{\ell_i}^{(m)}\\
&= \sum_{i=k+1}^p (D_{\ell_i}-ED) M_{\ell_i}^{(m)} I_{\{|D_{\ell_i}-ED|>\gamma a_{np}^2\}}\,,
\end{aligned}$$

then $\widetilde{D}_{(k+1)} = |D_{\ell_{k+1}} - ED| > \gamma a_{np}^2$ and therefore

$$N_n(\gamma) = \#\{i\le p : |D_i-ED| > \gamma a_{np}^2\} > k\,.$$

However, for fixed $\gamma>0$, in view of Theorem A.1, as $n\to\infty$,

$$P(N_n(\gamma)>k) \le k^{-1} EN_n(\gamma) = k^{-1} p\, P(|D-ED|>\gamma a_{np}^2) \sim k^{-1}\gamma^{-\alpha/2} np\, P(Z^2 > a_{np}^2)\to 0\,.$$

Thus we have proved that

$$\lim_{n\to\infty} P\Big(a_{np}^{-2}\Big\|\sum_{i=1}^{p}(D_i-ED)M_i^{(m)} I_{\{|D_i-ED|>\gamma a_{np}^2\}} - \sum_{i=1}^{k}(D_{\ell_i}-ED)M_{\ell_i}^{(m)} I_{\{|D_{\ell_i}-ED|>\gamma a_{np}^2\}}\Big\|_2>\delta\Big)=0\,.$$

Relation (5.21) is proved if we can show that for $\delta>0$,

$$\lim_{n\to\infty} P\Big(a_{np}^{-2}\Big\|\sum_{i=1}^{k}(D_{\ell_i}-ED) I_{\{|D_{\ell_i}-ED|\le\gamma a_{np}^2\}} M_{\ell_i}^{(m)}\Big\|_2>\delta\Big)\to 0\,, \tag{5.22}$$

as $\gamma\downarrow 0$. On the event $\widetilde{A}_n$ defined in (5.17), $\sum_{i=1}^k (D_{\ell_i}-ED) I_{\{|D_{\ell_i}-ED|\le\gamma a_{np}^2\}} M_{\ell_i}^{(m)}$ is block diagonal and therefore

$$a_{np}^{-2}\Big\|\sum_{i=1}^{k}(D_{\ell_i}-ED) I_{\{|D_{\ell_i}-ED|\le\gamma a_{np}^2\}} M_{\ell_i}^{(m)}\Big\|_2 = a_{np}^{-2}\, v_1^{(m)} \max_{i=1,\ldots,k} |D_{\ell_i}-ED| I_{\{|D_{\ell_i}-ED|\le\gamma a_{np}^2\}} \le c\, v_1\,\gamma\,.$$

We also observe that $P(A_n^c)\to 0$; see (5.18). Then (5.22) is immediate and the lemma is proved. □

**Step 4: Final argument. The case $\alpha\in(0,2)$.** On $A_n$ defined in (5.12), the matrix $\sum_{i=1}^k D_{L_i} M_{L_i}^{(m)}$ is block diagonal and has the non-negative eigenvalues $D_{L_i} v_j^{(m)} = D_{(i)} v_j^{(m)}$, $i=1,\ldots,k$, $j=1,\ldots,r_m$. The corresponding ordered values of them are denoted by $\delta_{(1)}^{(m)}\ge\cdots\ge\delta_{(p)}^{(m)}$. Combining Lemmas 5.1–5.3 with Weyl's inequality for the eigenvalues of Hermitian matrices (see Bhatia [3]) and recalling that $P(A_n^c)\to 0$ as $n\to\infty$, we have

$$\lim_{m\to\infty}\limsup_{n\to\infty} P\Big(a_{np}^{-2}\max_{i\le p}\Big|\lambda_{(i)}-\delta_{(i)}^{(m)}\Big|>\varepsilon\Big)=0\,,\quad \varepsilon>0\,.$$

Finally, we observe that

$$a_{np}^{-2}\max_{i\le p}\big|\delta_{(i)}^{(m)}-\delta_{(i)}\big|\le a_{np}^{-2}\max_{i\le p} D_i\ \max_{i\le p}\big|v_i^{(m)}-v_i\big| = o_P(1)\,,$$

since $a_{np}^{-2}\max_{i\le p} D_i \xrightarrow{d} \Gamma_1^{-\alpha/2}$ and $v_i^{(m)}\to v_i$ uniformly in $i$ because both sequences are monotone. This finishes the proof in the case $\alpha\in(0,2)$.

**The case $\alpha\in(2,4)$.** On $\widetilde{A}_n$ defined in (5.17), the matrix $\sum_{i=1}^k (D_{\ell_i}-ED) M_{\ell_1}^{(m)}$ is block diagonal and has the non-negative eigenvalues $(D_{\ell_i}-ED)v_j^{(m)}$, $i=1,\ldots,k$, $j=1,\ldots,r_m$. The corresponding ordered values are denoted by $\widetilde{\delta}_{(1)}^{(m)}\ge\cdots\ge\widetilde{\delta}_{(p)}^{(m)}$. Combining Lemmas 5.1–5.4 with Weyl's inequality and recalling that $P(\widetilde{A}_n^c)\to 0$, we have

$$\lim_{m\to\infty}\limsup_{n\to\infty} P\Big(a_{np}^{-2}\max_{i\le k}\Big|\widetilde{\lambda}_{(i)}-\widetilde{\delta}_{(i)}^{(m)}\Big|>\varepsilon\Big)=0\,,\quad \varepsilon>0\,.$$

As before, one observes that

$$a_{np}^{-2}\max_{i\le p}\big|\widetilde{\delta}_{(i)}^{(m)}-\widetilde{\delta}_{(i)}\big|\le a_{np}^{-2}\max_{i\le p}|D_i-ED|\ \max_{i\le p}\big|v_i^{(m)}-v_i\big| = o_P(1)\,,$$

This finishes the proof in the case $\alpha\in(2,4)$.

## Appendix A

A.1. **Large deviation results.** Let $(Z_i)$ be iid copies of $Z$ whose distribution satisfies (1.2) for some $\alpha > 0$. If $E|Z| < \infty$ also assume $EZ = 0$. Write

$$S_n = Z_1 + \cdots + Z_n\,, \quad n \ge 1,$$

and consider a sequence $(a_n)$ such that $P(|Z| > a_n) \sim n^{-1}$.

The following theorem can be found in Nagaev [13] and Cline and Hsing [5] for $\alpha > 2$ and $\alpha \le 2$, respectively; see also Denisov et al. [8].

**Theorem A.1.** *Under the assumptions on the iid sequence $(Z_t)$ given above the following relation holds*

$$\sup_{x \ge c_n} \left| \frac{P(S_n > x)}{nP(|Z| > x)} - p_+ \right| \to 0\,,$$

*where $(c_n)$ is any sequence satisfying $c_n/a_n \to \infty$ for $\alpha \le 2$ and $c_n \ge \sqrt{(\alpha-2)n\log n}$ for $\alpha > 2$.*

A.2. **A Karamata theorem for partial sums.** Assume that the conditions of the previous section hold.

**Lemma A.2.** *Let $(c_n)$ be the threshold sequences in Theorem A.1 for a given $\alpha > 0$, $\delta \in (0,1)$, and let $(d_n)$ be such that $d_n/c_n \to \infty$ for $\alpha > 2$ and $d_n = c_n$ for $\alpha \le 2$. Then we have for $\gamma > \alpha$, uniformly for $x \ge d_n$.*

$$E|x^{-1}S_n|^\gamma I_{\{\delta x \le |S_n| \le x\}} \sim \int_{\delta^\gamma}^1 y^{-\alpha/\gamma} dy\; P(|S_n| > x) \sim \int_{\delta^\gamma}^1 y^{-\alpha/\gamma}\, dy [n\, P(|Z| > x)]\,. \tag{A.1}$$

*Moreover, for every $\epsilon > 0$ there exists a constant $c > 0$ such that*

$$E|x^{-1}S_n|^\gamma I_{\{d_n \le |S_n| \le x\}} \le c\, n\, P(|Z| > x)\,, \quad x \ge d_n\,. \tag{A.2}$$

*Proof.* The last equivalence relation in (A.1) follows from Theorem A.1. We have for $x \ge d_n$,

$$\begin{aligned}
E|x^{-1}S_n|^\gamma I_{\{\delta \le |x^{-1}S_n| \le 1\}} &= \int_{\delta^\gamma}^1 P(|S_n/x| > y^{1/\gamma})\, dy \\
&= \int_{\delta^\gamma}^1 \frac{P(|S_n/x| > y^{1/\gamma})}{nP(|Z/x| > y^{1/\gamma})}\, [nP(|Z/x| > y^{1/\gamma})]\, dy \\
&\sim \int_{\delta^\gamma}^1 y^{-\alpha/\gamma}\, dy\, [nP(|Z| > x)]\,, \quad n \to \infty\,,
\end{aligned}$$

where we used Theorem A.1 and the uniform convergence theorem for regularly varying functions; see Bingham et al. [4]. Using the same approach, the fact that $xy^{1/\gamma} \ge d_n$ and Karamata's integral theorem,

$$\begin{aligned}
E|x^{-1}S_n|^\gamma I_{\{d_n/x \le |S_n/x| \le 1\}} &= \int_{(d_n/x)^\gamma}^1 P(|S_n/x| > y^{1/\gamma})\, dy \\
&\sim \int_{(d_n/x)^\gamma}^1 n\, P(|Z| > xy^{1/\gamma})\, dy \\
&\le n \int_0^1 P(|Z| > xy^{1/\gamma})\, dy \\
&\sim \frac{\gamma}{\gamma - \alpha} n\, P(|Z| > x)\,.
\end{aligned}$$

$\square$

A.3. **A point process convergence result.** Assume that the conditions of Section A.1 hold. Consider a sequence of iid copies $(S_n^{(t)})_{t=1,2,\ldots}$ of $S_n$ and the sequence of point processes

$$N_n = \sum_{t=1}^{p} \varepsilon_{a_{np}^{-1} S_n^{(t)}}, \quad n = 1, 2, \ldots,$$

for an integer sequence $p = p_n \to \infty$. We assume that the state space of the point processes $N_n$ is $\overline{\mathbb{R}}_0 = [\mathbb{R} \cup \{\pm\infty\}] \backslash \{0\}$.

**Lemma A.3.** *Assume $\alpha \in (0,2)$ and the conditions of Section A.1 on the iid sequence $(Z_t)$ and the normalizing sequence $(a_n)$. Then the limit relation $N_n \stackrel{d}{\to} N$ holds in the space of point measures on $\overline{\mathbb{R}}_0$ equipped with the vague topology (see* [15, 16]*) for a Poisson random measure $N$ with state space $\overline{\mathbb{R}}_0$ and intensity measure $\mu_\alpha(dx) = \alpha|x|^{-\alpha-1}(p_+ I_{\{x>0\}} + p_- I_{\{x<0\}})dx$.*

*Proof.* According to Resnick [15], Proposition 3.21, we need to show that $p\, P(a_{np}^{-1} S_n \in \cdot) \stackrel{v}{\to} \mu_\alpha$, where $\stackrel{v}{\to}$ denotes vague convergence of Radon measures on $\overline{\mathbb{R}}_0$. Observe that we have $a_{np}/a_n \to \infty$ as $n \to \infty$. This fact and $\alpha \in (0,2)$ allow one to apply Theorem A.1:

$$\frac{P(S_n > x a_{np})}{n\, P(|Z| > a_{np})} \to p_+ x^{-\alpha} \quad \text{and} \quad \frac{P(S_n \le -x a_{np})}{n\, P(|Z| > a_{np})} \to p_-\, x^{-\alpha}, \quad x > 0.$$

On the other hand, $n\, P(|Z| > a_{np}) \sim p^{-1}$ as $n \to \infty$. This proves the lemma. □

A.4. **Moment inequalities.** Let $Y_1, \ldots, Y_n$, $n \ge 1$ be independent random variables and define $\widetilde{S}_n = Y_1 + \cdots + Y_n$, $n \ge 1$. The following inequalities can be found e.g. in Petrov [14], Theorem 2.10 for $p \ge 2$, and on p. 82, 2.6.20, for $p \le 2$.

**Lemma A.4.** *Assume $E|Y_i|^p < \infty$, $i = 1, \ldots, p$, for some $p > 0$. If $p \le 1$ or $p \in [1,2]$ and $EY_i = 0$, $i = 1, \ldots, p$, then*

$$E|\widetilde{S}_n|^p \le c \sum_{i=1}^{n} E|Y_i|^p, \tag{A.3}$$

*If $p \ge 2$ and $EY_i = 0$, $i = 1, \ldots, p$, then*

$$E|\widetilde{S}_n|^p \le c\, n^{p/2-1} \sum_{i=1}^{n} E|Y_i|^p. \tag{A.4}$$

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