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Time-changed extremal process as a random sup measure

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A functional limit theorem for the partial maxima of a long memory stable sequence produces a limiting process that can be described as a β-power time change in the classical Fréchet extremal process, for β in a subinterval of the unit interval. Any such power time change in the extremal process for $0 < \beta < 1$ produces a process with stationary max-increments. This deceptively simple time change hides the much more delicate structure of the resulting process as a self-affine random sup measure. We uncover this structure and show that in a certain range of the parameters this random measure arises as a limit of the partial maxima of the same long memory stable sequence, but in a different space. These results open a way to construct a whole new class of self-similar Fréchet processes with stationary max-increments.

Keywords: extremal limit theorem; extremal process; heavy tails; random sup measure; stable process; stationary max-increments; self-similar process

1. Introduction

Let $(X_1, X_2, \ldots)$ be a stationary sequence of random variables, and let $M_n = \max_{1\le k\le n} X_k$, $n = 1, 2, \ldots$ be the sequence of its partial maxima. The limiting distributional behavior of the latter sequence is one of the major topics of interest in extreme value theory. We are particularly interested in the possible limits in a functional limit theorem of the form

$$\left(\frac{M_{\lfloor nt\rfloor} - b_n}{a_n}, t \ge 0\right) \Rightarrow \bigl(Y(t), t \ge 0\bigr), \tag{1.1}$$

for properly chosen sequences (a_n), (b_n). The weak convergence in (1.1) is typically in the space $D[0,\infty)$ with one of the usual Skorohod topologies on that space; see [1,22] and [26]. If the original sequence $(X_1, X_2, \ldots)$ is an i.i.d. sequence, then the only possible limit in (1.1) is *the extremal process*, the extreme value analog of the Lévy process; see [9].

The modern extreme value theory is interested in the case when the sequence $(X_1, X_2, \ldots)$ is stationary, but not necessarily independent. The potential clustering of the extremes in this case leads one to expect that new limits may arise in (1.1). Such new limits, however, have not been widely observed, and the dependence in the model has been typically found to be reflected in the limit via a linear time change (a slowdown), often connected to the *extremal index*, introduced originally in [10]. See, for example, [11], as well as the studies in [2,12,18] and [4]. One possible explanation for this is the known phenomenon that the operation of taking partial maxima tends to mitigate the effect of dependence in the original stationary sequence, and the dependent models considered above were, in a certain sense, not sufficiently strongly dependent.

Starting with a long range dependent sequence may make a difference, as was demonstrated by [15]. In that paper, the original sequence was (the absolute value of) a stationary symmetric α-stable process, $0 < \alpha < 2$, and the length of memory was quantified by a single parameter $0 < \beta < 1$. In the case $1/2 < \beta < 1$, it was shown that the limiting process in (1.1) can be represented in the form

$$Z_{\alpha,\beta}(t) = Z_\alpha\bigl(t^\beta\bigr), \qquad t \ge 0, \tag{1.2}$$

where $(Z_\alpha(t), t \ge 0)$ is the extremal (α-)Fréchet process.

The nonlinear power time change in (1.2) is both surprising and misleadingly simple. It is surprising because it is not immediately clear that such a change is compatible with a certain translation invariance the limiting process must have due to the stationarity of the original sequence. It is misleadingly simple because it hides a much more delicate structure. The main goal of this paper is to reveal that structure. We start by explaining exactly what we are looking for.

The stochastic processes in the left-hand side of (1.1) can be easily interpreted as *random sup measures* evaluated on a particular family of sets (those of the form $[0, t]$ for $t \ge 0$). If one does not restrict himself to that specific family of sets and, instead, looks at all Borel subsets of $[0, \infty)$, then it is possible to ask whether there is weak convergence in the appropriately defined space of random sup measures, and what might be the limiting random sup measures. See the discussion around (2.4) and the convergence result in Theorem 5.1. This is the approach taken in [14]. Completing the work published in [24] and [25], the authors provide a detailed description of the possible limits. They show that the limiting random sup measure must be *self-affine* (they refer to random sup measures as extremal processes, but we reserve this name for a different object).

As we will see in the sequel, if (1.1) can be stated in terms of weak convergence of a sequence of random sup measures, this would imply the finite-dimensional convergence part in the functional formulation of (1.1). Therefore, any limiting process Y that can be obtained as a limit in this case must be equal in distribution to the restriction of a random sup measure to the sets of the form $[0, t]$, $t \ge 0$. The convergence to the process $Z_{\alpha,\beta}$ established in [15] was not established in the sense of weak convergence of a sequence of random sup measures, and one of our tasks in this paper is to fill this gap and prove the above convergence. Recall, however, that the convergence in [15] was established only for $0 < \alpha < 2$ (by necessity, since α-stable processes do not exist outside of this range) and $1/2 < \beta < 1$. The nonlinear time change in (1.2) is, however, well defined for all $\alpha > 0$ and $0 < \beta < 1$, and leads to a process $Z_{\alpha,\beta}$ that is self-similar and has stationary max-increments. Our second task in this paper is to prove that the process $Z_{\alpha,\beta}$ can, for *all values of its parameters*, be extended to a random sup measure and elucidate the structure of the resulting random sup measure. The key result is Corollary 4.4 below. The structure we obtain is of interest on its own right. It is constructed based on a certain random closed set possessing appropriate scaling and translation invariance properties. Extending this approach to other random sets and other ways of handling these random sets may potentially lead to a construction of new classes of self-similar processes with stationary max-increments and of random sup measures. This is important both theoretically, and may be useful in applications.

This paper is organized as follows. In the next section, we will define precisely the notions discussed somewhat informally above and introduce the required technical background. Section 3 contains a discussion of the dynamics of the stationary sequence considered in this paper. It is based on a null recurrent Markov chain. In Section 4, we will prove that the process $Z_{\alpha,\beta}$ can be extended to a random sup measure and construct explicitly such an extension. In Section 5, we show that the convergence result of [15] holds, in a special case of a Markovian ergodic system, also in the space SM of sup measures. Finally, in Section 6 we present one of the possible extensions of the present work.

2. Background

An extremal process $(Y(t), t \geq 0)$ can be viewed as an analog of a Lévy motion when the operation of summation is replaced by the operation of taking the maximum. The one-dimensional marginal distribution of a Lévy process at time 1 can be an arbitrary infinitely divisible distribution on $\mathbb{R}$; *any* one-dimensional distribution is infinitely divisible with respect to the operation of taking the maximum. Hence the one-dimensional marginal distribution of an extremal process at time 1 can be any distribution on $[0, \infty)$; the restriction to the nonnegative half-line being necessitated by the fact that, by convention, an extremal process, analogously to a Lévy process, starts at the origin at time zero. If F is the c.d.f. of a probability distribution on $[0, \infty)$, then the finite-dimensional distributions of an extremal process with distribution F at time 1 can be defined by

$$\begin{aligned}&\big(Y(t_1), Y(t_2), \dots, Y(t_n)\big)\\&\quad \stackrel{d}{=} \big(X_{t_1}^{(1)}, \max\big(X_{t_1}^{(1)}, X_{t_2-t_1}^{(2)}\big), \dots, \max\big(X_{t_1}^{(1)}, X_{t_2-t_1}^{(2)}, \dots, X_{t_n-t_{n-1}}^{(n)}\big)\big)\end{aligned} \tag{2.1}$$

for all $n \geq 1$ and $0 \leq t_1 < t_2 < \dots < t_n$. The different random variables in the right-hand side of (2.1) are independent, with $X_t^{(k)}$ having the c.d.f. F^t for $t > 0$. In this paper, we deal with the α-Fréchet extremal process, for which

$$F(x) = F_{\alpha,\sigma}(x) = \exp\{-\sigma^\alpha x^{-\alpha}\}, \qquad x > 0, \tag{2.2}$$

the Fréchet law with the tail index $\alpha > 0$ and the scale $\sigma > 0$. A stochastic process $(Y(t), t \in T)$ (on an arbitrary parameter space T) is called a Fréchet process if for all $n \geq 1$, $a_1, \dots, a_n > 0$ and $t_1, \dots, t_n \in T$, the weighted maximum $\max_{1\leq j\leq n} a_j Y(t_j)$ has a Fréchet law as in (2.2). Obviously, the Fréchet extremal process is an example of a Fréchet process, but there are many Fréchet processes on $[0, \infty)$ different from the Fréchet extremal process; the process $Z_{\alpha,\beta}$ in (1.2) is one such process.

A stochastic process $(Y(t), t \geq 0)$ is called self-similar with exponent H of self-similarity if for any $c > 0$

$$\big(Y(ct), t \geq 0\big) \stackrel{d}{=} \big(c^H Y(t), t \geq 0\big)$$

in the sense of equality of finite-dimensional distributions. A stochastic process $(Y(t), t \geq 0)$ is said to have stationary max-increments if for every $r \geq 0$, there exists, perhaps on an enlarged probability space, a stochastic process $(Y^{(r)}(t), t \ge 0)$ such that

$$\begin{cases} \big(Y^{(r)}(t), t \ge 0\big) \overset{d}{=} \big(Y(t), t \ge 0\big), \\ \big(Y(t+r), t \ge 0\big) \overset{d}{=} \big(Y(r) \vee Y^{(r)}(t), t \ge 0\big), \end{cases} \tag{2.3}$$

with $a \vee b = \max(a, b)$; see [15]. This notion is an analog of the usual notion of a process with stationary increments (see, e.g., [3] and [20]) suitable for the situation where the operation of summation is replaced by the operation of taking the maximum. It follows from Theorem 3.2 in [15] that only self-similar processes with stationary max-increments can be obtained as limits in the functional convergence scheme (1.1) with $b_n \equiv 0$.

We switch next to a short overview of random sup measures. The reader is referred to [14] for full details. Let $\mathcal{G}$ be the collection of open subsets of $[0, \infty)$. We call a map $m : \mathcal{G} \to [0, \infty]$ a sup measure (on $[0, \infty)$) if $m(\varnothing) = 0$ and

$$m\Big(\bigcup_{r \in R} G_r\Big) = \sup_{r \in R} m(G_r)$$

for an arbitrary collection $(G_r, r \in R)$ of open sets. In general, a sup measure can take values in any closed subinterval of $[-\infty, \infty]$, not necessarily in $[0, \infty]$, but we will consider, for simplicity, only the nonnegative case in the sequel, and restrict ourselves to the maxima of nonnegative random variables as well.

The *sup derivative* of a sup measure is a function $[0, \infty) \to [0, \infty]$ defined by

$$d^{\vee} m(t) = \inf_{G \ni t} m(G), \qquad t \ge 0.$$

It is automatically an upper semicontinuous function. Conversely, for any function $f : [0, \infty) \to [0, \infty]$ the *sup integral* of f is a sup measure defined by

$$i^{\vee} f(G) = \sup_{t \in G} f(t), \qquad G \in \mathcal{G},$$

with $i^{\vee} f(\varnothing) = 0$ by convention. It is always true that $m = i^{\vee} d^{\vee} m$ for any sup measure m, but the statement $f = d^{\vee} i^{\vee} f$ is true only for upper semicontinuous functions f. A sup measure has a canonical extension to all subsets of $[0, \infty)$ via

$$m(B) = \sup_{t \in B} d^{\vee} m(t).$$

On the space SM of sup measures, one can introduce a topology, called the *sup vague topology* that makes SM a compact metric space. In this topology, a sequence (m_n) of sup measures converges to a sup measure m if both

$$\limsup_{n \to \infty} m_n(K) \le m(K) \qquad \text{for every compact } K$$

and

$$\liminf_{n \to \infty} m_n(G) \ge m(G) \qquad \text{for every open } G.$$

A random sup measure is a measurable map from a probability space into the space SM equipped with the Borel σ-field generated by the sup vague topology.

The convergence scheme (1.1) has a natural version in terms of random sup measures. Starting with a stationary sequence $\mathbf{X} = (X_1, X_2, \dots)$ of nonnegative random variables, one can define for any set $B \subseteq [0, \infty)$

$$M_n(\mathbf{X})(B) = \max_{k:k/n \in B} X_k. \tag{2.4}$$

Then for any $a_n > 0$, $M_n(\mathbf{X})/a_n$ is a random sup measure, and [14] characterizes all possible limiting random sup measures in a statement of the form

$$\frac{M_n(\mathbf{X})}{a_n} \Rightarrow M \tag{2.5}$$

for some sequence (a_n). The convergence is weak convergence in the space SM equipped with the sup vague topology. Theorem 6.1 in [14] shows that any limiting random sup measure M must be both stationary and self-similar, that is,

$$M(a + \cdot) \stackrel{d}{=} M \quad \text{and} \quad a^{-H} M(a\cdot) \stackrel{d}{=} M \qquad \text{for all } a > 0 \tag{2.6}$$

for some exponent H of self-similarity. In fact, the results of [14] allow for a shift (b_n) as in (1.1), in which case the power scaling a^{-H} in (2.6) is, generally, replaced by the scaling of the form $\delta^{-\log a}$, where δ is an affine transformation. In the context of the present paper, this additional generality does not play a role.

Starting with a stationary and self-similar random sup measure M, one defines a stochastic process by

$$Y(t) = M\bigl((0, t]\bigr), \qquad t \ge 0. \tag{2.7}$$

Then the self-similarity property of the random sup measure M immediately implies the self-similarity property of the stochastic process Y, with the same exponent of self-similarity. Furthermore, the stationarity of the random sup measure M implies that the stochastic process Y has stationary max-increments; indeed, for $r \ge 0$ one can simply take

$$Y^{(r)}(t) = M\bigl((r, r+t]\bigr), \qquad t \ge 0.$$

Whether or not any self-similar process with stationary max-increments can be constructed in this way or, in other words, whether or not such a process can be extended, perhaps on an extended probability space, to a stationary and self-similar random sup measure remains, to the best of our knowledge, an open question. We do show that the process $Z_{\alpha,\beta}$ in (1.2) has such an extension.

3. The Markov chain dynamics

The stationary sequence we will consider in Section 5 is a symmetric α-stable (SαS) sequence, whose dynamics is driven by a certain Markov chain. Specifically, consider an irreducible null recurrent Markov chain $(Y_n, n \geq 0)$ defined on an infinite countable state space $\mathbb{S}$ with transition matrix (p_{ij}). Fix an arbitrary state $i_0 \in \mathbb{S}$, and let $(\pi_i, i \in \mathbb{S})$ be the unique invariant measure of the Markov chain with $\pi_{i_0} = 1$. Note that (π_i) is necessarily an infinite measure.

Define a σ-finite and infinite measure on $(E, \mathcal{E}) = (\mathbb{S}^{\mathbb{N}}, \mathcal{B}(\mathbb{S}^{\mathbb{N}}))$ by

$$\mu(B) = \sum_{i \in \mathbb{S}} \pi_i P_i(B), \qquad B \in \mathcal{E},$$

where $P_i(\cdot)$ denotes the probability law of (Y_n) starting in state $i \in \mathbb{S}$. Clearly, the usual left shift operator on $\mathbb{S}^{\mathbb{N}}$

$$T(x_0, x_1, \ldots) = (x_1, x_2, \ldots)$$

preserves the measure μ. Since the Markov chain is irreducible and null recurrent, T is conservative and ergodic (see [6]).

Consider the set $A = \{x \in \mathbb{S}^{\mathbb{N}} : x_0 = i_0\}$ with the fixed state $i_0 \in \mathbb{S}$ chosen above. Let

$$\varphi_A(x) = \min\bigl\{n \geq 1 : T^n x \in A\bigr\}, \qquad x \in \mathbb{S}^{\mathbb{N}}$$

be the first entrance time, and assume that

$$\sum_{k=1}^{n} P_{i_0}(\varphi_A \geq k) \in RV_\beta,$$

the set of regularly varying sequences with exponent β of regular variation, for $\beta \in (0, 1)$. By the Tauberian theorem for power series (see, e.g., [5]), this is equivalent to assuming that

$$P_{i_0}(\varphi_A \geq k) \in RV_{\beta-1}. \tag{3.1}$$

There are many natural examples of Markov chains with this property. Probably, the simplest example is obtained by taking $\mathbb{S} = \{0, 1, 2, \ldots\}$ and letting the transition probabilities satisfy $p_{i,i-1} = 1$ for $i \geq 1$, with $(p_{0,j}, j = 0, 1, 2, \ldots)$ being an arbitrary probability distribution satisfying

$$\sum_{j=k}^{\infty} p_{0,j} \in RV_{\beta-1}, \qquad k \to \infty.$$

Let $f \in L^\infty(\mu)$ be a nonnegative function on $\mathbb{S}^{\mathbb{N}}$ supported by A. Define for $0 < \alpha < 2$

$$b_n = \left(\int_E \max_{1 \leq k \leq n} \bigl(f \circ T^k(x)\bigr)^\alpha \mu(dx) \right)^{1/\alpha}, \qquad n = 1, 2, \ldots. \tag{3.2}$$

The sequence (b_n) plays an important part in [15], and it will play an important role in this paper as well. If we define *the wandering rate sequence* by

$$w_n = \mu\bigl(\bigl\{x \in \mathbb{S}^{\mathbb{N}} : x_j = i_0 \text{ for some } j = 0, 1, \ldots, n\bigr\}\bigr), \qquad n = 1, 2, \ldots,$$

then, clearly, $w_n \sim \mu(\varphi_A \le n)$ as $n \to \infty$. We know by Theorem 4.1 in [15] that

$$\lim_{n\to\infty} \frac{b_n^\alpha}{w_n} = \|f\|_\infty. \tag{3.3}$$

Furthermore, it follows from Lemma 3.3 in [17] that

$$w_n \sim \sum_{k=1}^{n} P_{i_0}(\varphi_A \ge k) \in RV_\beta.$$

The above setup allows us to define a stationary symmetric α-stable (SαS) sequence by

$$X_n = \int_E f \circ T^n(x)\, dM(x), \qquad n = 1, 2, \dots, \tag{3.4}$$

where M is a SαS random measure on $(E, \mathcal{E})$ with control measure μ. See [21] for details on α-stable random measures and integrals with respect to these measures. This is a long range dependent sequence, and the parameter β of the Markov chain determined just how long the memory is; see [15,16]. Section 5 of the present paper discusses an extremal limit theorem for this sequence.

4. Random sup measure structure

In this section, we prove a limit theorem, and the limit in this theorem is a stationary and self-similar random sup measure whose restrictions to the intervals of the type $(0, t]$, $t \ge 0$, as in (2.7) is distributionally equal to the process $Z_{\alpha,\beta}$ in (1.2). This result is also a major step toward the extension of the main result in [15] to the setup in (2.5) of weak convergence in the space of sup measures of normalized partial maxima of the absolute values of a SαS sequence. The extension itself is formally proved in the next section. We emphasize that the discussion in this section applies to all $0 < \beta < 1$.

We introduce first some additional setup. Let $L_{1-\beta}$ be the standard $(1-\beta)$-stable subordinator, that is, an increasing Lévy process such that

$$Ee^{-\theta L_{1-\beta}(t)} = e^{-t\theta^{1-\beta}} \qquad \text{for } \theta \ge 0 \text{ and } t \ge 0.$$

Let

$$R_\beta = \overline{\{L_{1-\beta}(t), t \ge 0\}} \subset [0, \infty) \tag{4.1}$$

be (the closure of) the range of the subordinator. It has several very attractive properties as a random closed set, described in the following proposition. We equip the space $\mathcal{J}$ of closed subsets of $[0, \infty)$ with the usual Fell topology (see [13]), and the Borel σ-field generated by that topology. We will use some basic facts about measurability of $\mathcal{J}$-valued maps and equality of measures on $\mathcal{J}$; these are stated in the proof of the proposition below. It is always sufficient to consider "hitting" open sets, and among the latter it is sufficient to consider finite unions of open intervals.

Proposition 4.1. *Let $\beta \in (0,1)$ and R_β be the range* (4.1) *of the standard* $(1-\beta)$*-stable subordinator $L_{1-\beta}$ defined on some probability space $(\Omega, \mathcal{F}, P)$. Then*:

(a) *R_β is a random closed subset of* $[0,\infty)$.

(b) *For any $a > 0$, $aR_\beta \overset{d}{=} R_\beta$ as random closed sets.*

(c) *Let μ_β be a measure on $(0,\infty)$ given by $\mu_\beta(dx) = \beta x^{\beta-1}\,dx$, $x>0$, and let $\kappa_\beta = (\mu_\beta \times P) \circ H^{-1}$, where $H : (0,\infty) \times \Omega \to \mathcal{J}$ is defined by $H(x,\omega) = R_\beta(\omega) + x$. Then for any $r > 0$ the measure κ_β is invariant under the shift map $G_r : \mathcal{J} \to \mathcal{J}$ given by*

$$G_r(F) = F \cap [r,\infty) - r.$$

Proof. For part (a), we need to check that for any open $G \subseteq [0,\infty)$, the set

$$\{\omega \in \Omega : R_\beta(\omega) \cap G \neq \varnothing\}$$

is in $\mathcal{F}$. By the right continuity of sample paths of the subordinator, the same set can be written in the form

$$\{\omega \in \Omega : L_{1-\beta}(r) \in G \text{ for some rational } r\}.$$

Now the measurability is obvious.

Part (b) is a consequence of the self-similarity of the subordinator. Indeed, it is enough to check that for any open $G \subseteq [0,\infty)$

$$P(R_\beta \cap G \neq \varnothing) = P(aR_\beta \cap G \neq \varnothing).$$

However, by the self-similarity,

$$\begin{aligned} P(R_\beta \cap G \neq \varnothing) &= P\big(L_{1-\beta}(r) \in G \text{ for some rational } r\big) \\ &= P\big(aL_{1-\beta}\big(a^{-(1-\beta)}r\big) \in G \text{ for some rational } r\big) = P(aR_\beta \cap G \neq \varnothing), \end{aligned}$$

as required.

For part (c) it is enough to check that for any finite collection of disjoint intervals, $0 < b_1 < c_1 < b_2 < c_2 < \cdots < b_n < c_n < \infty$

$$\begin{aligned} &\kappa_\beta\left(\left\{F \in \mathcal{J} : F \cap \bigcup_{j=1}^{n} (b_j, c_j) \neq \varnothing\right\}\right) \\ &\quad = \kappa_\beta\left(\left\{F \in \mathcal{J} : F \cap \bigcup_{j=1}^{n} (b_j + r, c_j + r) \neq \varnothing\right\}\right); \end{aligned} \tag{4.2}$$

see Example 1.29 in [13]. A simple inductive argument together with the strong Markov property of the subordinator shows that it is enough to prove (4.2) for the case of a single interval. That is, one has to check that for any $0 < b < c < \infty$,

$$\kappa_\beta\big(\{F \in \mathcal{J} : F \cap (b,c) \neq \varnothing\}\big) = \kappa_\beta\big(\{F \in \mathcal{J} : F \cap (b+r, c+r) \neq \varnothing\}\big). \tag{4.3}$$

For $h > 0$, let

$$\delta_h = \inf\{y : y \in R_\beta \cap [h, \infty)\} - h$$

be the overshoot of the level h by the subordinator $L_{1-\beta}$. Then (4.3) can be restated in the form

$$\begin{aligned}&\int_0^b \beta x^{\beta-1} P(\delta_{b-x} < c-b)\,dx + (c^\beta - b^\beta)\\&\quad= \int_0^{b+r} \beta x^{\beta-1} P(\delta_{b+r-x} < c-b)\,dx + ((c+r)^\beta - (b+r)^\beta).\end{aligned}$$

The overshoot δ_h is known to have a density with respect to the Lebesgue measure, given by

$$f_h(y) = \frac{\sin(\pi(1-\beta))}{\pi} h^{1-\beta}(y+h)^{-1} y^{\beta-1}, \qquad y > 0; \tag{4.4}$$

see, for example, Exercise 5.6 in [8], and checking the required identity is a matter of somewhat tedious but still elementary calculations. □

In the notation of Section 3, we define for $n = 1, 2, \dots$ and $x \in E = \mathbb{S}^{\mathbb{N}}$ a sup measure on $[0, \infty)$ by

$$m_n(B; x) = \max_{k: k/n \in B} f \circ T^k(x), \qquad B \subseteq [0, \infty). \tag{4.5}$$

The main result of this section will be stated in terms of weak convergence of a sequence of finite-dimensional random vectors. Its significance will go well beyond that weak convergence, as we will describe in the sequel. Let $0 \le t_1 < t_1' \le \dots \le t_m < t_m' < \infty$ be fixed points, $m \ge 1$. For $n = 1, 2, \dots$ let $Y^{(n)} = (Y_1^{(n)}, \dots, Y_m^{(n)})$ be an m-dimensional Fréchet random vector satisfying

$$P(Y_1^{(n)} \le \lambda_1, \dots, Y_m^{(n)} \le \lambda_m) = \exp\left\{-\int_E \bigvee_{i=1}^m \lambda_i^{-\alpha} m_n((t_i, t_i'); x)^\alpha \mu(dx)\right\}, \tag{4.6}$$

for $\lambda_j > 0$, $j = 1, \dots, m$; see, for example, [23] for details on Fréchet random vectors and processes.

Theorem 4.2. *Let $0 < \beta < 1$. The sequence of random vectors $(b_n^{-1} Y^{(n)})$ converges weakly in $\mathbb{R}^m$ to a Fréchet random vector $Y^* = (Y_1^*, \dots, Y_m^*)$ such that*

$$\begin{aligned}&P(Y_1^* \le \lambda_1, \dots, Y_m^* \le \lambda_m)\\&\quad= \exp\left\{-E'\left(\int_0^\infty \bigvee_{i=1}^m \lambda_i^{-\alpha} \mathbf{1}((R_\beta + x) \cap (t_i, t_i') \ne \varnothing)\beta x^{\beta-1}\,dx\right)\right\}\end{aligned} \tag{4.7}$$

for $\lambda_j > 0$, $j = 1, \dots, m$, where R_β is the range (4.1) of a $(1-\beta)$-stable subordinator defined on some probability space $(\Omega', \mathcal{F}', P')$.

We postpone proving the theorem and discuss first its significance. Define

$$W_{\alpha,\beta}(A) = {}^{e}\!\!\int_{(0,\infty)\times\Omega'} \mathbf{1}\big(\big(R_\beta(\omega') + x\big) \cap A \neq \varnothing\big) M\big(dx, d\omega'\big), \qquad A \subseteq [0,\infty), \text{ Borel.} \tag{4.8}$$

The integral in (4.8) is the extremal integral with respect to a Fréchet random sup measure M on $(0,\infty)\times\Omega'$, where $(\Omega', \mathcal{F}', P')$ is some probability space. We refer the reader to [23] for details. The control measure of M is $m = \mu_\beta \times P'$, where μ_β is defined in part (c) of Proposition 4.1. It is evident that $W_{\alpha,\beta}(A) < \infty$ a.s. for any bounded Borel set A. We claim that a version of $W_{\alpha,\beta}$ is a random sup measure on $[0,\infty)$. The necessity of taking an appropriate version stems from the usual phenomenon, that the extremal integral is defined separately for each set A, with a corresponding A-dependent exceptional set.

Let $N_{\alpha,\beta}$ be a Poisson random measure on $(0,\infty)^2$ with the mean measure

$$\alpha x^{-(\alpha+1)}\, dx\, \beta y^{\beta-1}\, dy, \qquad x, y > 0.$$

Let $((U_i, V_i))$ be a measurable enumeration of the points of $N_{\alpha,\beta}$. Let, further, $(R_\beta^{(i)})$ be i.i.d. copies of the range of the $(1-\beta)$-stable subordinator, independent of the Poisson random measure $N_{\alpha,\beta}$. Then a version of $W_{\alpha,\beta}$ is given by

$$\hat{W}_{\alpha,\beta}(A) = \bigvee_{i=1}^{\infty} U_i \mathbf{1}\big(\big(R_\beta^{(i)} + V_i\big) \cap A \neq \varnothing\big), \qquad A \subseteq [0,\infty), \text{ Borel;} \tag{4.9}$$

see [23]. It is interesting to note that, since the origin belongs, with probability 1, to the range of the subordinator, evaluating (4.9) on sets of the form $A = [0,t]$, $0 \le t \le 1$, reduces this representation to the more standard representation of the process $Z_{\alpha,\beta}$ in (1.2). See (3.8) in [15].

It is clear that $\hat{W}_{\alpha,\beta}$ is a random sup measure on $[0,\infty)$. In fact,

$$d^{\vee}\hat{W}_{\alpha,\beta}(t) = \begin{cases} U_i, & \text{if } t \in R_\beta^{(i)} + V_i, \text{ some } i = 1, 2, \ldots, \\ 0, & \text{otherwise.} \end{cases} \tag{4.10}$$

Even though it is $\hat{W}_{\alpha,\beta}$ that takes values in the space of sup measures, we will slightly abuse the terminology and refer to $W_{\alpha,\beta}$ itself a random sup measure.

Proposition 4.3. *For any $\beta \in (0,1)$, the random sup measure $W_{\alpha,\beta}$ is stationary and self-similar with exponent $H = \beta/\alpha$ in the sense of* (2.6).

Proof. Both statements can be read off (4.10). Indeed, the pairs $(U_i, (R_\beta^{(i)} + V_i))$ form a Poisson random measure on $(0,\infty) \times \mathcal{J}$ and, by part (c) of Proposition 4.1, the mean measure of this Poisson random measure is unaffected by the transformations G_r applied to the random set dimension. This implies the law of the random upper semicontinuous function $d^{\vee}\hat{W}_{\alpha,\beta}$ is shift invariant, hence stationarity of $W_{\alpha,\beta}$.

For the self-similarity, note that replacing t by t/a, $a > 0$ in (4.10) is equivalent to replacing $R_\beta^{(i)}$ by $aR_\beta^{(i)}$ and V_i by aV_i. By part (b) of Proposition 4.1, the former action does not change the law of a random closed set, while it is elementary to check that the law of the Poisson random measure on $(0,\infty)^2$ with points $((U_i, aV_i))$ is the same as the law of the Poisson random measure on the same space with the points $((a^{\beta/\alpha}U_i, V_i))$. Hence, the self-similarity of $W_{\alpha,\beta}$ with $H=\beta/\alpha$. □

Returning now to the result in Theorem 4.2, note that it can be restated in the form

$$b_n^{-1}\big(Y_1^{(n)},\dots,Y_m^{(n)}\big) \Rightarrow \big(W_{\alpha,\beta}\big((t_1,t_1')\big),\dots,W_{\alpha,\beta}\big((t_m,t_m')\big)\big) \qquad \text{as } n\to\infty.$$

In particular, if we choose $t_i = t'_{i-1}$, $i=1,\dots,m$, with $t_1=0$ and an arbitrary t_{m+1}, and define

$$Z_i^{(n)} = \max_{j=1,\dots,i} Y_j^{(n)}, \qquad i=1,\dots,m,$$

then

$$\begin{aligned}\big(b_n^{-1}Z_i^{(n)}, i=1,\dots,m\big) &\Rightarrow \Big(\max_{j=1,\dots,i} W_{\alpha,\beta}\big((t_j,t_{j+1})\big), i=1,\dots,m\Big)\\ &= \big(W_{\alpha,\beta}\big((0,t_{i+1})\big), i=1,\dots,m\big).\end{aligned} \tag{4.11}$$

However, as a part of the argument in [15] it was established that

$$\big(b_n^{-1}Z_i^{(n)}, i=1,\dots,m\big) \Rightarrow \big(Z_{\alpha,\beta}(t_{i+1}), i=1,\dots,m\big),$$

with $Z_{\alpha,\beta}$ as in (1.2); this is (4.7) in [15]. This leads to the immediate conclusion, stated in the following corollary.

Corollary 4.4. *For any $\beta\in(0,1)$, the time-changed extremal Fréchet process satisfies*

$$\big(Z_{\alpha,\beta}(t), t\ge 0\big) \stackrel{d}{=} \big(W_{\alpha,\beta}\big((0,t]\big), t\ge 0\big)$$

and, hence, is a restriction of the stationary and self-similar random sup measure $W_{\alpha,\beta}$ (to the intervals $(0,t]$, $t\ge 0$).

We continue with a preliminary result, needed for the proof of Theorem 4.2, which may also be of independent interest.

Proposition 4.5. *Let $0<\gamma<1$, and $(Y_1,Y_2,\dots)$ be i.i.d. nonnegative random variables such that $P(Y_1>y)$ is regularly varying with exponent $-\gamma$. Let $S_0=0$ and $S_n=Y_1+\cdots+Y_n$ for $n=1,2,\dots$ be the corresponding partial sums. For $\theta>0$ define a random sup measure on $[0,\infty)$ by*

$$M^{(Y;\theta)}(G) = \mathbf{1}(S_n\in\theta G \text{ for some } n=0,1,\dots)$$

$G\subseteq[0,\infty)$, open. Then

$$M^{(Y;\theta)} \Rightarrow_{\theta\to\infty} M^{(\gamma)}$$

in the space SM equipped with the sup vague topology, where

$$M^{(\gamma)}(G) = \mathbf{1}(R_{1-\gamma} \cap G \neq \varnothing).$$

Proof. It is enough to prove that for any finite collection of intervals $(a_i, b_i), i = 1, \dots, m$ with $0 < a_i < b_i < \infty, i = 1, \dots, m$ we have

$$\begin{aligned} &P\bigl(\text{for each } i = 1, \dots, m, S_j/\theta \in (a_i, b_i) \text{ for some } j = 1, 2, \dots\bigr) \\ &\quad \to P\bigl(\text{for each } i = 1, \dots, m, R_{1-\gamma} \cap (a_i, b_i) \neq \varnothing\bigr) \end{aligned} \tag{4.12}$$

as $\theta \to \infty$. If we let $a(\theta) = (P(Y_1 > \theta))^{-1}$, a regularly varying function with exponent γ, then the probability in the left-hand side of (4.12) can be rewritten as

$$P\bigl(\text{for each } i = 1, \dots, m, S_{\lfloor ta(\theta)\rfloor}/\theta \in (a_i, b_i) \text{ for some } t \geq 0\bigr). \tag{4.13}$$

By the invariance principle,

$$(S_{\lfloor ta(\theta)\rfloor}/\theta, t \geq 0) \Rightarrow_{\theta\to\infty} \bigl(L_\gamma(t), t \geq 0\bigr) \tag{4.14}$$

weakly in the J_1-topology in the space $D[0, \infty)$, where L_γ is the standard γ-stable subordinator; see, for example, [7]. If we denote by $D_+^\uparrow[0, \infty)$ the set of all nonnegative nondecreasing functions in $D[0, \infty)$ vanishing at $t = 0$, then $D_+^\uparrow[0, \infty)$ is, clearly, a closed set in the J_1-topology, so the weak convergence in (4.14) also takes places in the J_1-topology relativized to $D_+^\uparrow[0, \infty)$.

For a function $\varphi \in D_+^\uparrow[0, \infty)$, let

$$R_\varphi = \overline{\bigl\{\varphi(t), t \geq 0\bigr\}}$$

be the closure of its range. Notice that

$$R_\varphi = \Bigl(\bigcup_{t>0} \bigl(\varphi(t-), \varphi(t)\bigr)\Bigr)^c,$$

which makes it evident that for any $0 < a < b < \infty$ the set

$$\bigl\{\varphi \in D_+^\uparrow[0, \infty) : R_\varphi \cap [a, b] = \varnothing\bigr\}$$

is open in the J_1-topology, hence measurable. Therefore, the set

$$\bigl\{\varphi \in D_+^\uparrow[0, \infty) : R_\varphi \cap (a, b) \neq \varnothing\bigr\} = \bigcup_{k=1}^\infty \bigl\{\varphi \in D_+^\uparrow[0, \infty) : R_\varphi \cap [a + 1/k, b - 1/k] \neq \varnothing\bigr\}$$

is measurable as well and, hence, so is the set

$$\bigl\{\varphi \in D_+^\uparrow[0, \infty) : \text{for each } i = 1, \dots, m, R_\varphi \cap (a_i, b_i) \neq \varnothing\bigr\}.$$

Therefore, the desired conclusion (4.12) will follow from (4.13) and the invariance principle (4.14) once we check that the measurable function on $D_+^{\uparrow}[0,\infty)$ defined by

$$J(\varphi) = \mathbf{1}\big(R_\varphi \cap (a_i, b_i) \neq \varnothing \text{ for each } i = 1, \dots, m\big)$$

is a.s. continuous with respect to the law of L_γ on $D_+^{\uparrow}[0,\infty)$. To see this, let

$$B_1 = \big\{\varphi \in D_+^{\uparrow}[0,\infty) : \text{for each } i = 1, \dots, m \text{ there is } t_i \text{ such that } \varphi(t_i) \in (a_i, b_i)\big\}$$

and

$$B_2 = \big\{\varphi \in D_+^{\uparrow}[0,\infty) : \text{for some } i = 1, \dots, m \text{ there is } t_i \text{ such that } (a_i, b_i) \subseteq \big(\varphi(t_i-), \varphi(t_i)\big)\big\}.$$

Both sets are open in the J_1-topology on $D_+^{\uparrow}[0,\infty)$, and $J(\varphi) = 1$ on B_1 and $J(\varphi) = 0$ on B_2. Now the a.s. continuity of the function J follows from the fact that

$$P(L_\gamma \in B_1 \cup B_2) = 1,$$

since a stable subordinator does not hit fixed points. □

Remark 4.6. It follows immediately from Proposition 4.5 that we also have weak convergence in the space of closed subsets of $[0,\infty)$. Specifically, the random closed set $\theta^{-1}\{S_n, n = 0, 1, \dots\}$ converges weakly, as $\theta \to \infty$, to the random closed set $R_{1-\gamma}$.

Proof of Theorem 4.2. We will prove that

$$\begin{aligned} &\frac{\int_E \min_{i=1,\dots,m} m_n((t_i, t_i'); x)^\alpha \mu(dx)}{\int_E \max_{1\le k\le n}(f \circ T^n(x))^\alpha \mu(dx)} \\ &\quad \to \int_0^\infty \beta x^{\beta-1} P'\big((R_\beta + x) \cap \big(t_i, t_i'\big) \neq \varnothing \text{ for each } i = 1, \dots, m\big)\, dx \end{aligned} \tag{4.15}$$

as $n \to \infty$. The reason this will suffice for the proof of the theorem is that, by the inclusion–exclusion formula, the expression in the exponent in the right-hand side of (4.7) can be written as a finite linear combination of terms of the form of the right-hand side of (4.15) (with different collections of intervals in each term). More specifically, we can write, for a fixed $x > 0$,

$$\begin{aligned} &E'\left(\bigvee_{i=1}^m \lambda_i^{-\alpha} \mathbf{1}\big((R_\beta + x) \cap \big(t_i, t_i'\big) \neq \varnothing\big)\right) \\ &\quad = \int_0^\infty P'\big((R_\beta + x) \cap \big(t_i, t_i'\big) \neq \varnothing \text{ for some } i \text{ such that } \lambda_i^{-\alpha} > u\big)\, du \end{aligned}$$

and apply the inclusion–exclusion formula to the probability of the union inside the integral. A similar relation exists between the left-hand side of (4.15) and the distribution of $(b_n^{-1} Y^{(n)})$.

An additional simplification that we may and will introduce is that of assuming that f is constant on A. Indeed, it follows immediately from the ergodicity that both the numerator and the denominator in the left-hand side of (4.15) do not change asymptotically if we replace f by $\|f\|_\infty \mathbf{1}_A$; see (4.2) in [15]. With this simplification, (4.15) reduces to the following statement: as $n \to \infty$,

$$\begin{aligned}
&\frac{1}{w_n}\mu\left(\bigcap_{i=1}^{m}\{x_k = i_0 \text{ for some } k \text{ with } t_i < k/n < t_i'\}\right) \\
&\quad\to \int_0^\infty \beta x^{\beta-1} P'\big((R_\beta + x) \cap (t_i, t_i') \neq \varnothing \text{ for each } i = 1, \ldots, m\big)\, dx.
\end{aligned} \tag{4.16}$$

Note that we have used (3.3) in translating (4.15) into the form (4.16).

We introduce the notation $A_0 = A$, $A_k = A^c \cap \{\varphi_A = k\}$ for $k \geq 1$. Let $(Y_1, Y_2, \ldots)$ be a sequence of i.i.d. $\mathbb{N}$-valued random variables defined on some probability space $(\Omega', \mathcal{F}', P')$ such that $P'(Y_1 = k) = P_{i_0}(\varphi_A = k)$, $k = 1, 2, \ldots$. By our assumption, the probability tail $P(Y_1 > y)$ is regularly varying with exponent $-(1-\beta)$. With $S_0 = 0$ and $S_j = Y_1 + \cdots + Y_j$ for $j = 1, 2, \ldots$ we have

$$\begin{aligned}
&\mu\left(\bigcap_{i=1}^{m}\{x_k = i_0 \text{ for some } k \text{ with } t_i < k/n < t_i'\}\right) \\
&\quad = \sum_{l: l/n \leq t_1} \mu(A_l) P'\big(\text{for each } i = 1, \ldots, m, S_j \in (nt_i - l, nt_i' - l) \text{ for some } j = 0, 1, \ldots\big) \\
&\qquad + \sum_{l: t_1 < l/n < t_1'} \mu(A_l) P'\big(\text{for each } i = 2, \ldots, m, S_j \in (nt_i - l, nt_i' - l) \text{ for some } j = 0, 1, \ldots\big) \\
&\quad := D_n^{(1)} + D_n^{(2)}.
\end{aligned}$$

It is enough to prove that

$$\lim_{n\to\infty} \frac{1}{w_n} D_n^{(1)} = \int_0^{t_1} \beta x^{\beta-1} P'\big((R_\beta + x) \cap (t_i, t_i') \neq \varnothing \text{ for each } i = 1, \ldots, m\big)\, dx \tag{4.17}$$

and

$$\lim_{n\to\infty} \frac{1}{w_n} D_n^{(2)} = \int_{t_1}^{t_1'} \beta x^{\beta-1} P'\big((R_\beta + x) \cap (t_i, t_i') \neq \varnothing \text{ for each } i = 1, \ldots, m\big)\, dx. \tag{4.18}$$

We will prove (4.17), and (4.18) can be proved in the same way. Let K be a large positive integer, and $\varepsilon > 0$ a small number. For each integer $1 \leq d \leq (1-\epsilon)K$, and each $l : t_1(d-1)/K \leq l/n < t_1 d/K$, we have

$$\begin{aligned}
&P'\big(\text{for each } i = 1, \ldots, m, S_j \in (nt_i - l, nt_i' - l) \text{ for some } j = 0, 1, \ldots\big) \\
&\quad \leq P'\big(\text{for each } i = 1, \ldots, m, S_j \in (nt_i - nt_1 d/K, nt_i' - nt_1(d-1)/K) \text{ for some } j = 0, 1, \ldots\big)
\end{aligned}$$

$$\to P'\big(\text{for each } i = 1, \dots, m,\ R_\beta \cap \big(t_i - t_1 d/K, t_i' - t_1(d-1)/K\big) \neq \varnothing\big)$$

as $n \to \infty$, by Proposition 4.5. Therefore,

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{w_n} D_n^{(1)} \\
&\quad \le \sum_{d=1}^{\lfloor (1-\epsilon)K \rfloor} \Bigg[\limsup_{n\to\infty} \frac{\sum_{l: t_1(d-1)/K \le l/n < t_1 d/K} \mu(A_l)}{w_n} \\
&\qquad \times P'\big(\text{for each } i = 1, \dots, m,\ R_\beta \cap \big(t_i - t_1 d/K, t_i' - t_1(d-1)/K\big) \neq \varnothing\big) \Bigg] \\
&\qquad + \limsup_{n\to\infty} \frac{\sum_{l: t_1 \lfloor (1-\epsilon)K \rfloor / K \le l/n \le t_1} \mu(A_l)}{w_n}.
\end{aligned}$$

Since for any $a > 0$,

$$\sum_{l=1}^{na} \mu(A_l) \sim w_{\lfloor na \rfloor} \qquad \text{as } n \to \infty,$$

and the wandering sequence (w_n) is regularly varying with exponent β, we conclude that

$$\begin{aligned}
\limsup_{n\to\infty} \frac{\sum_{l: t_1(d-1)/K \le l/n < t_1 d/K} \mu(A_l)}{w_n} &= \limsup_{n\to\infty} \frac{w_{\lfloor n t_1 d/K \rfloor} - w_{\lfloor n t_1 (d-1)/K \rfloor}}{w_n} \\
&= \frac{t_1^\beta}{K^\beta} \big(d^\beta - (d-1)^\beta\big)
\end{aligned}$$

for $1 \le d \le (1-\epsilon)K$ and, similarly,

$$\limsup_{n\to\infty} \frac{\sum_{l: t_1 \lfloor (1-\epsilon)K \rfloor / K \le l/n \le t_1} \mu(A_l)}{w_n} = t_1^\beta \left[1 - \left(\frac{\lfloor (1-\varepsilon)K \rfloor}{K} \right)^\beta \right].$$

Therefore,

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{w_n} D_n^{(1)} \\
&\quad \le \int_0^{(1-\varepsilon)t_1} \beta x^{\beta-1} P'\big(R_\beta \cap \big(t_i - a_K(x), t_i' - b_K(x)\big) \neq \varnothing \text{ for each } i = 1, \dots, m\big)\, dx \\
&\qquad + t_1^\beta \left[1 - \left(\frac{\lfloor (1-\varepsilon)K \rfloor}{K} \right)^\beta \right],
\end{aligned}$$

where $a_K(x) = t_1 d/K$ and $b_K(x) = t_1(d-1)/K$ if $t_1(d-1)/K \le x < t_1 d/K$ for $1 \le d \le (1-\epsilon)K$. Since

$$\mathbf{1}\big(R_\beta \cap (a_k, b_k) \neq \varnothing\big) \to \mathbf{1}\big(R_\beta \cap (a, b) \neq \varnothing\big)$$

a.s. if $a_k \to a$ and $b_k \to b$, we can let $K \to \infty$ and then $\varepsilon \to 0$ to conclude that

$$\limsup_{n\to\infty} \frac{1}{w_n} D_n^{(1)} \leq \int_0^{t_1} \beta x^{\beta-1} P'\bigl(R_\beta \cap \bigl(t_i - x, t_i' - x\bigr) \neq \varnothing \text{ for each } i = 1, \ldots, m\bigr)\, dx. \tag{4.19}$$

We can obtain a lower bound matching (4.19) in a similar way. Indeed, for each integer $1 \leq d \leq (1-\epsilon)K$, and each $l : t_1(d-1)/K \leq l/n < t_1 d/K$ as above, we have

$$\begin{aligned}
&P'\bigl(\text{for each } i = 1, \ldots, m, S_j \in \bigl(nt_i - l, nt_i' - l\bigr) \text{ for some } j = 0, 1, \ldots\bigr) \\
&\quad \geq P'\bigl(\text{for each } i = 1, \ldots, m, S_j \in \bigl(nt_i - nt_1(d-1)/K, nt_i' - nt_1 d/K\bigr) \text{ for some } j = 0, 1, \ldots\bigr) \\
&\quad \to P'\bigl(\text{for each } i = 1, \ldots, m, R_\beta \cap \bigl(t_i - t_1(d-1)/K, t_i' - t_1 d/K\bigr) \neq \varnothing\bigr)
\end{aligned}$$

as $n \to \infty$, by Proposition 4.5, and we proceed as before. This gives a lower bound complementing (4.19), so we have proved that

$$\lim_{n\to\infty} \frac{1}{w_n} D_n^{(1)} = \int_0^{t_1} \beta x^{\beta-1} P'\bigl(R_\beta \cap \bigl(t_i - x, t_i' - x\bigr) \neq \varnothing \text{ for each } i = 1, \ldots, m\bigr)\, dx.$$

This is, of course, (4.17). □

5. Convergence in the space SM

Let $\mathbf{X} = (X_1, X_2, \ldots)$ be the stationary SαS process defined by (3.4). The following theorem is a partial extension of Theorem 4.1 in [15] to weak convergence in the space of sup measures. In its statement, we use the usual tail constant of an α-stable random variable given by

$$C_\alpha = \left(\int_0^\infty x^{-\alpha} \sin x \, dx\right)^{-1} = \begin{cases} (1-\alpha)/\bigl(\Gamma(2-\alpha)\cos(\pi\alpha/2)\bigr), & \text{if } \alpha \neq 1, \\ 2/\pi, & \text{if } \alpha = 1; \end{cases}$$

see [21].

Theorem 5.1. *For $n = 1, 2, \ldots$ define a random sup measure $M_n(|\mathbf{X}|)$ on $[0, \infty)$ by (2.4), with $|\mathbf{X}| = (|X_1|, |X_2|, \ldots)$. Let (b_n) be given by (3.2). If $1/2 < \beta < 1$, then*

$$\frac{1}{b_n} M_n\bigl(|\mathbf{X}|\bigr) \Rightarrow C_\alpha^{1/\alpha} W_{\alpha,\beta} \qquad \text{as } n \to \infty \tag{5.1}$$

in the sup vague topology in the space SM.

Proof. The weak convergence in the space SM will be established if we show that for any $0 \leq t_1 < t_1' \leq \cdots \leq t_m < t_m' < \infty$,

$$\bigl(b_n^{-1} M_n\bigl(|\mathbf{X}|\bigr)\bigl(\bigl(t_1, t_1'\bigr)\bigr), \ldots, b_n^{-1} M_n\bigl(|\mathbf{X}|\bigr)\bigl(\bigl(t_m, t_m'\bigr)\bigr)\bigr) \Rightarrow C_\alpha^{1/\alpha}\bigl(W_{\alpha,\beta}\bigl(\bigl(t_1, t_1'\bigr)\bigr), \ldots, W_{\alpha,\beta}\bigl(\bigl(t_m, t_m'\bigr)\bigr)\bigr)$$

as $n \to \infty$ (see Section 12.7 in [25]). For simplicity of notation, we will assume that $t'_m \le 1$. Our goal is, then, to show that

$$\left(\frac{1}{b_n}\max_{nt_1<k<nt'_1}|X_k|,\ldots,\frac{1}{b_n}\max_{nt_m<k<nt'_m}|X_k|\right) \Rightarrow C_\alpha^{1/\alpha}\bigl(W_{\alpha,\beta}\bigl((t_1,t'_1)\bigr),\ldots,W_{\alpha,\beta}\bigl((t_m,t'_m)\bigr)\bigr) \tag{5.2}$$

as $n \to \infty$.

We proceed in the manner similar to that adopted in [15], and use a series representation of the SαS sequence $(X_1, X_2, \ldots)$. Specifically, we have

$$(X_k, k=1,\ldots,n) \stackrel{d}{=} \left(b_n C_\alpha^{1/\alpha}\sum_{j=1}^\infty \epsilon_j \Gamma_j^{-1/\alpha}\frac{f\circ T^k(U_j^{(n)})}{\max_{1\le i\le n} f\circ T^i(U_j^{(n)})}, k=1,\ldots,n\right). \tag{5.3}$$

In the right-hand side, (ϵ_j) are i.i.d. Rademacher random variables (symmetric random variables with values ± 1), (Γ_j) are the arrival times of a unit rate Poisson process on $(0,\infty)$, and $(U_j^{(n)})$ are i.i.d. E-valued random variables with the common law η_n defined by

$$\frac{d\eta_n}{d\mu}(x) = \frac{1}{b_n^\alpha}\max_{1\le k\le n} f\circ T^k(x)^\alpha, \qquad x \in E. \tag{5.4}$$

The three sequences (ϵ_j), (Γ_j) and $(U_j^{(n)})$ are independent. We refer the reader to Section 3.10 of [21] for details on series representations of α-stable processes. We will prove that for any $\lambda_i > 0, i = 1,\ldots,m$ and $0 < \delta < 1$,

$$\begin{aligned} &P\Bigl(b_n^{-1}\max_{nt_i<k<nt'_i}|X_k| > \lambda_i, i=1,\ldots,m\Bigr) \\ &\quad\le P\left(C_\alpha^{1/\alpha}\bigvee_{j=1}^\infty \Gamma_j^{-1/\alpha}\frac{\max_{nt_i<k<nt'_i} f\circ T^k(U_j^{(n)})}{\max_{1\le k\le n} f\circ T^k(U_j^{(n)})} > \lambda_i(1-\delta), i=1,\ldots,m\right) + o(1) \end{aligned} \tag{5.5}$$

and that

$$\begin{aligned} &P\Bigl(b_n^{-1}\max_{nt_i<k<nt'_i}|X_k| > \lambda_i, i=1,\ldots,m\Bigr) \\ &\quad\ge P\left(C_\alpha^{1/\alpha}\bigvee_{j=1}^\infty \Gamma_j^{-1/\alpha}\frac{\max_{nt_i<k<nt'_i} f\circ T^k(U_j^{(n)})}{\max_{1\le k\le n} f\circ T^k(U_j^{(n)})} > \lambda_i(1+\delta), i=1,\ldots,m\right) + o(1) \end{aligned} \tag{5.6}$$

as $n \to \infty$. Before doing so, we will make a few simple observations. Let

$$V_i^{(n)} = \bigvee_{j=1}^\infty \Gamma_j^{-1/\alpha}\frac{\max_{nt_i<k<nt'_i} f\circ T^k(U_j^{(n)})}{\max_{1\le k\le n} f\circ T^k(U_j^{(n)})}, \qquad i=1,\ldots,m.$$

Since the points in $\mathbb{R}^m$ given by

$$\left(\Gamma_j^{-1/\alpha} \frac{\max_{nt_i < k < nt_i'} f \circ T^k(U_j^{(n)})}{\max_{1 \le k \le n} f \circ T^k(U_j^{(n)})}, i = 1, \ldots, m\right), \qquad j = 1, 2, \ldots$$

form a Poisson random measure on $\mathbb{R}^m$, say, N_P, for $\lambda_i > 0$, $i = 1, \ldots, m$ we can write

$$\begin{aligned} P\big(V_1^{(n)} \le \lambda_1, \ldots, V_m^{(n)} \le \lambda_m\big) &= P\big(N_P\big(D(\lambda_1, \ldots, \lambda_m) = 0\big)\big) \\ &= \exp\big\{-E\big(N_P\big(D(\lambda_1, \ldots, \lambda_m)\big)\big)\big\}, \end{aligned}$$

where

$$D(\lambda_1, \ldots, \lambda_m) = \big\{(z_1, \ldots, z_m) : z_i > \lambda_i \text{ for some } i = 1, \ldots, m\big\}.$$

Evaluating the expectation, we conclude that, in the notation of (4.5),

$$P\big(V_1^{(n)} \le \lambda_1, \ldots, V_m^{(n)} \le \lambda_m\big) = \exp\left\{-b_n^{-\alpha} \int_E \bigvee_{i=1}^m \lambda_i^{-\alpha} m_n\big((t_i, t_i'); x\big)^\alpha \mu(dx)\right\}.$$

By (4.6), this shows that, in the notation of Theorem 4.2,

$$\big(V_1^{(n)}, \ldots, V_m^{(n)}\big) \stackrel{d}{=} \big(b_n^{-1} Y_1^{(n)}, \ldots, b_n^{-1} Y_m^{(n)}\big).$$

Now Theorem 4.2 along with the discussion following the statement of that theorem, and the continuity of the Fréchet distribution show that (5.2) and, hence, the claim of the present theorem, will follow once we prove (5.5) and (5.6). The two statements can be proved in a very similar way, so we only prove (5.5).

Once again, we proceed as in [15]. Choose constants $K \in \mathbb{N}$ and $0 < \epsilon < 1$ such that both

$$K + 1 > \frac{4}{\alpha} \quad \text{and} \quad \delta - \epsilon K > 0.$$

Then

$$\begin{aligned} &P\Big(b_n^{-1} \max_{nt_i < k < nt_i'} |X_k| > \lambda_i, i = 1, \ldots, m\Big) \\ &\quad \le P\left(C_\alpha^{1/\alpha} \bigvee_{j=1}^\infty \Gamma_j^{-1/\alpha} \frac{\max_{nt_i < k < nt_i'} f \circ T^k(U_j^{(n)})}{\max_{1 \le k \le n} f \circ T^k(U_j^{(n)})} > \lambda_i(1-\delta), i = 1, \ldots, m\right) \\ &\qquad + \varphi_n\Big(C_\alpha^{-1/\alpha} \epsilon \min_{1 \le i \le m} \lambda_i\Big) + \sum_{i=1}^m \psi_n\big(\lambda_i, t_i, t_i'\big), \end{aligned}$$

where

$$\varphi_n(\eta) = P\left(\bigcup_{k=1}^n \left\{\Gamma_j^{-1/\alpha} \frac{f \circ T^k(U_j^{(n)})}{\max_{1 \le i \le n} f \circ T^i(U_j^{(n)})} > \eta \text{ for at least 2 different } j = 1, 2, \ldots\right\}\right),$$

$\eta > 0$, and for $t < t'$,

$$\psi_n(\lambda, t, t') = P\Bigg(C_\alpha^{1/\alpha} \max_{nt<k<nt'} \Bigg|\sum_{j=1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \frac{f \circ T^k(U_j^{(n)})}{\max_{1\le i\le n} f \circ T^i(U_j^{(n)})}\Bigg| > \lambda,$$

$$C_\alpha^{1/\alpha} \bigvee_{j=1}^{\infty} \Gamma_j^{-1/\alpha} \frac{\max_{nt<k<nt'} f \circ T^k(U_j^{(n)})}{\max_{1\le k\le n} f \circ T^k(U_j^{(n)})} \le \lambda(1-\delta), \text{ and for each } l = 1, \dots, n,$$

$$C_\alpha^{1/\alpha} \Gamma_j^{-1/\alpha} \frac{f \circ T^l(U_j^{(n)})}{\max_{1\le i\le n} f \circ T^i(U_j^{(n)})} > \epsilon\lambda \text{ for at most one } j = 1, 2, \dots\Bigg).$$

Due to the assumption $1/2 < \beta < 1$, it follows that

$$\varphi_n\Big(C_\alpha^{-1/\alpha} \epsilon \min_{1\le i\le m} \lambda_i\Big) \to 0$$

as $n \to \infty$; see [19]. Therefore, the proof will be completed once we check that for all $\lambda > 0$ and $0 \le t < t' \le 1$,

$$\psi_n(\lambda, t, t') \to 0$$

This, however, can be checked in exactly the same way as (4.10) in [15]. □

6. Other processes based on the range of the subordinator

The distributional representation of the time-changed extremal process (1.2) in Corollary 4.4 can be stated in the form

$$Z_{\alpha,\beta}(t) = {}^{e}\!\!\int_{(0,\infty)\times\Omega'} \mathbf{1}\big(\big(R_\beta(\omega') + x\big) \cap (0, t] \ne \varnothing\big) M\big(dx, d\omega'\big), \qquad t \ge 0. \tag{6.1}$$

The self-similarity property of the process and the stationarity of its max-increments can be traced to the scaling and shift invariance properties of the range of the subordinator described in Proposition 4.1. These properties can be used to construct other self-similar processes with stationary max-increments, in the manner similar to the way scaling and shift invariance properties of the real line have been used to construct integral representations of Gaussian and stable self-similar processes with stationary increments such as fractional Brownian and stable motions; see, for example, [21] and [3].

In this section, we describe one family of self-similar processes with stationary max-increments, which can be viewed as an extension of the process in (6.1). Other processes can be constructed; we postpone a more general discussion to a later work.

For $0 \le s < t$, we define a function $j_{s,t} : \mathcal{J} \to [0, \infty]$ by

$$j_{s,t}(F) = \sup\big\{b - a : s < a < t, a, b \in F, (a, b) \cap F = \varnothing\big\},$$

the "length of the longest empty space within F beginning between s and t". The function $j_{s,t}$ is continuous, hence measurable, on $\mathcal{J}$. Set also $j_{s,s}(F) \equiv 0$. Let

$$0 < \gamma < (1-\beta)/\alpha, \tag{6.2}$$

and define

$$\begin{aligned} Z_{\alpha,\beta,\gamma}(t) = {}^{e}\!\!\int_{(0,\infty)\times\Omega'} &\big[\mathbf{1}\big(\big(R_\beta(\omega')+x\big)\cap(0,t]\neq\varnothing\big)j_{0,t}\big(R_\beta(\omega')+x\big)\big]^\gamma \\ &\times M\big(dx,d\omega'\big), \qquad t\geq 0. \end{aligned} \tag{6.3}$$

It follows from (4.4) that

$$E'\bigg(\int_0^\infty \big[\mathbf{1}\big((R_\beta+x)\cap(0,t]\neq\varnothing\big)j_{0,t}(R_\beta+x)\big]^{\gamma\alpha}\beta x^{\beta-1}\,dx\bigg) < \infty$$

for γ satisfying (6.2). Therefore, (6.3) presents a well-defined Fréchet process. We claim that this process is H-self-similar with

$$H = \gamma + \beta/\alpha$$

and has stationary max-increments.

To check stationarity of max-increments, let $r > 0$ and define

$$Z^{(r)}_{\alpha,\beta,\gamma}(t) = {}^{e}\!\!\int_{(0,\infty)\times\Omega'} \big[\mathbf{1}\big(\big(R_\beta(\omega')+x\big)\cap(r,r+t]\neq\varnothing\big)j_{r,r+t}\big(R_\beta(\omega')+x\big)\big]^\gamma M\big(dx,d\omega'\big), \qquad t\geq 0.$$

Trivially, for every $t \geq 0$ we have

$$Z_{\alpha,\beta,\gamma}(r) \vee Z^{(r)}_{\alpha,\beta,\gamma}(t) = Z_{\alpha,\beta,\gamma}(r+t)$$

with probability 1, and it follows from part (c) of Proposition 4.1 that

$$\big(Z^{(r)}_{\alpha,\beta,\gamma}(t), t\geq 0\big) \stackrel{d}{=} \big(Z_{\alpha,\beta,\gamma}(t), t\geq 0\big).$$

Hence, stationarity of max-increments. Finally, we check the property of self-similarity. Let $t_j > 0, \lambda_j > 0$, $j = 1, \ldots, m$. Then

$$P\big(Z_{\alpha,\beta,\gamma}(t_j) \leq \lambda_j,\, j=1,\ldots,m\big) = \exp\big\{-I(t_1,\ldots,t_m;\lambda_1,\ldots,\lambda_m)\big\},$$

where

$$\begin{aligned} &I(t_1,\ldots,t_m;\lambda_1,\ldots,\lambda_m) \\ &\quad = E'\bigg(\int_0^\infty \beta x^{\beta-1} \max_{k=1,\ldots,m} \lambda_k^{-\alpha}\big[\mathbf{1}\big(\big(R_\beta(\omega')+x\big)\cap(0,t_k]\neq\varnothing\big)j_{0,t_k}\big(R_\beta(\omega')+x\big)\big]^{\gamma\alpha}\,dx\bigg). \end{aligned}$$

Therefore, the property of self-similarity will follow once we check that for any $c > 0$,

$$I(ct_1, \dots, ct_m; \lambda_1, \dots, \lambda_m) = I\bigl(t_1, \dots, t_m; c^{-H}\lambda_1, \dots, c^{-H}\lambda_m\bigr).$$

This is, however immediate, since by using first part (b) of Proposition 4.1 and, next, changing the variable of integration to $y = x/c$ we have

$$\begin{aligned}
&I(ct_1, \dots, ct_m; \lambda_1, \dots, \lambda_m) \\
&\quad = E'\biggl(\int_0^\infty \beta x^{\beta-1} \max_{k=1,\dots,m} \bigl\{\lambda_k^{-\alpha}\bigl[\mathbf{1}\bigl(\bigl(cR_\beta\bigl(\omega'\bigr) + x\bigr) \cap (0, ct_k] \neq \varnothing\bigr) \\
&\qquad \times \sup\bigl\{b - a : 0 < a < ct_j, a, b \in cR_\beta\bigl(\omega'\bigr) + x, (a, b) \cap cR_\beta\bigl(\omega'\bigr) + x = \varnothing\bigr\}\bigr]^{\alpha\gamma}\bigr\}\, dx\biggr) \\
&\quad = c^{\beta+\alpha\gamma} E'\biggl(\int_0^\infty \beta x^{\beta-1} \max_{k=1,\dots,m} \bigl\{\lambda_k^{-\alpha}\bigl[\mathbf{1}\bigl(\bigl(R_\beta\bigl(\omega'\bigr) + x\bigr) \cap (0, t_k] \neq \varnothing\bigr) \\
&\qquad \times \sup\bigl\{b - a : 0 < a < t_k, a, b \in R_\beta\bigl(\omega'\bigr) + x, (a, b) \cap R_\beta\bigl(\omega'\bigr) + x = \varnothing\bigr\}\bigr]^{\alpha\gamma}\bigr\}\, dx\biggr) \\
&\quad = I\bigl(t_1, \dots, t_m; c^{-H}\lambda_1, \dots, c^{-H}\lambda_m\bigr),
\end{aligned}$$

as required.

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