EXTREME VALUES OF THE UNIFORM ORDER 1 AUTOREGRESSIVE PROCESSES AND MISSING OBSERVATIONS

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Abstract. We investigate partial maxima of the uniform $AR(1)$ processes with parameter $r \geqslant 2$. Positively and negatively correlated processes are considered. New limit theorems for maxima in complete and incomplete samples are obtained.

1. Introduction

We consider an apparently simple stationary stochastic process with standard uniform marginals. The process is defined as a first order autoregressive ($AR(1)$) model

$$X_n = \alpha X_{n-1} + \varepsilon_n, \quad n \geqslant 1, \tag{1.1}$$

where $(\varepsilon_n)_{n\geqslant 1}$ is an i.i.d. sequence of innovations, independent of the initial value X_0. We will consider two different $AR(1)$ processes, each one parametrized by an integer $r \geqslant 2$. In both cases the initial state X_0 is taken to be a standard uniform random variable.

Positively correlated uniform $AR(1)$ processes are defined by (1.1) with $\alpha = 1/r$, where $r \geqslant 2$ is an integer. In this case a generic noise variable ε_n takes one of the r discrete values $\{0, 1/r, 2/r, \ldots, (r-1)/r\}$ with equal probabilities $1/r$. These processes were introduced by Chernick (1981). Obviously, a positively correlated uniform $AR(1)$ process is stationary, and each X_n has the standard uniform distribution.

Negatively correlated uniform $AR(1)$ processes are also defined by (1.1), but now $\alpha = -1/r$, with $r \geqslant 2$ still an integer. This time a generic noise variable ε_n takes one of the r discrete values $\{1/r, 2/r, \ldots, (r-1)/r, 1\}$ with equal probabilities $1/r$. These processes were introduced by Chernick and Davis (1982).

The extreme values of the positively and negatively correlated uniform $AR(1)$ processes are interesting, and have attracted attention because its extremes cluster in a somewhat unusual way. Recall that, marginally, the standard uniform distribution is in the Weibull domain of attraction, and for $u_n = 1 - x/n$ with

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$x > 0$ we have $P\{\max_{1\leqslant i\leqslant n} U_i \leqslant u_n\} \to e^{-x}$ as $n \to \infty$, if $U_1, U_2, \ldots$ are i.i.d. standard uniform random variables. For this sequence (u_n) both positively and negatively correlated uniform $AR(1)$ processes satisfy the $D(u_n)$ condition of Leadbetter (1974). This was shown by Chernick (1981) and Chernick and Davis (1982). If these processes also satisfied the $D'(u_n)$ condition of Leadbetter (1974), then the extremes of these processes would not cluster, and the partial maxima of these processes would satisfy a limit theorem with the same normalization and the same limit as the i.i.d. standard uniform sequence. This is, however, not true, and condition $D'(u_n)$ fails for the positively and negatively correlated uniform $AR(1)$ processes. In fact, it was shown by Chernick (1981) that the partial maximum $M_n = \max\{X_1, \ldots, X_n\}$ of the positively correlated uniform $AR(1)$ process satisfies $P\{M_n \leqslant 1 - x/n\} \to \exp\{-(1-r^{-1})x\}$ as $n \to \infty$, while Chernick and Davis (1982) showed that for the negatively correlated uniform $AR(1)$ processes one has $P\{M_n \leqslant 1 - x/n\} \to \exp\{-(1-r^{-2})x\}$ as $n \to \infty$. In particular, the extremes of the two processes cluster, and the extremal index of the two processes is equal to $1 - r^{-1}$ and $1 - r^{-2}$ in the positive correlated and negatively correlated cases, correspondingly. See Leadbetter (1983).

Even among processes whose extremes cluster, the uniform $AR(1)$ processes may be unusual. We explain this point briefly. The stationary process $Y_n = (1 - X_n)^{-1}$, $n = 0, 1, \ldots$ has standard Pareto (1) marginals, and, more generally, multivariate regularly varying distributions with exponent 1 of regular variation. Therefore, *the spectral tail process* is well defined; it can be obtained (extending first the $\mathbf{Y}$ process in law to a stationary process $(\ldots, Y_{-1}, Y_0, Y_1, Y_2, \ldots)$ indexed by $\mathbb{Z}$) by

$$P\left[\left(\frac{Y_n}{Y_0},\, n \in \mathbb{Z}\right) \in \cdot \,\middle|\, |Y_0| > y\right] \Rightarrow P\left[\left(T_n,\, n \in \mathbb{Z}\right) \in \cdot\right]$$

as $y \to \infty$; see Basrak and Segers (2009). Since the process $\mathbf{Y}$ is a Markov chain, one can expect, in accordance to the theory of Segers (2007) and Janssen and Segers (2014) that the tail process is, itself, a Markov chain of a particular type, the so-called back-and-forth tail chain. This is, indeed, the case if the process $\mathbf{Y}$ corresponds to the positively correlated uniform $AR(1)$ process. It is not difficult to check that in this situation one has, in law,

$$T_n = \begin{cases} r^n & \text{if } n \leqslant 0, \\ \prod_{j=1}^{n} A_j^{(r)} & \text{if } n > 0, \end{cases}$$

where $A_1^{(r)}, A_2^{(r)}, \ldots$ are i.i.d. random variables, $P(A_1^{(r)} = r) = 1 - P(A_1^{(r)} = 0) = 1/r$. On the other hand, if the process $\mathbf{Y}$ corresponds to the negatively correlated uniform $AR(1)$ process, then the spectral tail process is not even a Markov chain. It is not difficult to check that in the above notation we can now write, in law,

$$T_n = \begin{cases} r^n & \text{if } n \leqslant 0 \text{ is even}, \\ \prod_{j=1}^{n/2} A_j^{(r^2)} & \text{if } n > 0 \text{ is even}, \\ 0 & \text{if } n \text{ is odd}. \end{cases}$$

This happens because, if the process $\mathbf{X}$ is a negatively correlated uniform $AR(1)$ process, then the process $\mathbf{Y}$ does not satisfy Condition 2.2 of Segers (2007) or of Janssen and Segers (2014).

We are interested in studying the extreme value theory of the uniform $AR(1)$ processes in incomplete samples. This means that only some of the observations of the process of interest are registered, and one studies the extremal behaviour of the registered observations. Questions of this type are of obvious interest in situations where more than one natural frequency of observations exists (for example, in finance), or where one may be interested in less frequent observations of the state of a physical system (e.g. annual) than the natural frequency (e.g. daily observations). The observation scheme could more generally be driven by a mechanism independent of the process of interest. Studying extreme value theory for incomplete samples has venerable history, starting, probably, with Mittal (1978), and a number of new results appeared more recently, such as Scotto (2005), Hall and Hüsler (2006), Hall and Scotto (2008). Most of the previous work concentrated on the cases where the registered observations were either equally spaced, or were registered in a periodic manner. For us periodically registered observations provide one of the examples, and we will discuss it in the sequel.

In order to obtain a fuller picture of the extremes of a partially registered random sequence, it is useful to understand the joint behavior of the maxima of both fully and partially registered observations. Let $\mathbf{X} = (X_n)$ be a stationary process, and let $c_n \in \{0,1\}$, $n = 1, 2, \ldots$, be a deterministic sequence defining the registration: the observation X_n of the process is registered if $c_n = 1$ and is not registered otherwise. Then

$$M_n = \max_{1 \leqslant i \leqslant n} X_i \quad \text{and} \quad \widetilde{M}_n = \max_{1 \leqslant i \leqslant n:\, c_i = 1} X_i$$

are the two partial maxima of interest. Clearly, M_n is bigger of $\widetilde{M}_n$, and the partial maximum of the nonregistered observations, so, in particular, $M_n \geqslant \widetilde{M}_n$ for every n. We will assume existence of an asymptotic sampling frequency

$$p = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} c_j \in (0,1]\,. \tag{1.2}$$

Under very general conditions it was shown in Mladenović and Piterbarg (2006) that if the extremes of the process $\mathbf{X}$ do not cluster, i.e. if the $D'(u_n)$ condition of Leadbetter (1974) holds for the process $\mathbf{X}$ for an appropriate for a domain of attraction sequence (u_n), then the partial maxima of the registered and nonregistered observations are asymptotically independent, which fully determines the joint limiting law of the appropriately normalized pair $\big(\widetilde{M}_n, M_n\big)$. In particular, that limiting law exists, and is completely determined by the asymptotic frequency p in (1.2). This result, of course, does not apply in the case of the uniform $AR(1)$ processes. In fact, it was shown by examples in Mladenović (2009) and Mladenović and Živadinović (2015) that neither asymptotic independence of the partial maxima of the registered and nonregistered observations holds, nor is the limiting law of $\big(\widetilde{M}_n, M_n\big)$ determined by the asymptotic frequency p.

It is the purpose of this paper to give a general solution to the problem of the joint asymptotic behaviour of the properly normalized pair $\big(\widetilde{M}_n, M_n\big)$. We will provide sufficient conditions for existence of the limiting distribution, explain what features of the sampling sequence (c_n) determine the limiting distribution, and describe the form of the limiting distribution. Our main results are stated in Section 2, which also provides a number of examples and a discussion. The proofs are given in Section 3. Finally, Section 4 contains certain estimates concerning the application of the $D(u_n, v_n)$ condition in our arguments.

2. The limit theorems for the partial maxima

The marginal distributions of the uniform $AR(1)$ processes dictate the manner in which the partial maxima should be normalized. Specifically, we will study the limiting behaviour of the random vector $\big(n(1-\widetilde{M}_n), n(1-M_n)\big)$, try to determine existence and the shape of the limit

$$G(x,y) = \lim_{n\to\infty} P\left\{\widetilde{M}_n \leqslant 1-\frac{x}{n},\, M_n \leqslant 1-\frac{y}{n}\right\}, \quad x,y>0\,. \tag{2.1}$$

Notice that, if $0<x\leqslant y$, then

$$\begin{aligned} & P\left\{\widetilde{M}_n \leqslant 1-\frac{x}{n},\, M_n \leqslant 1-\frac{y}{n}\right\} = P\left\{M_n \leqslant 1-\frac{y}{n}\right\} \\ & \to \begin{cases} \exp\left\{-(1-r^{-1})y\right\} & \text{for positively dependent } \mathbf{X}, \\ \exp\left\{-(1-r^{-2})y\right\} & \text{for negatively dependent } \mathbf{X}, \end{cases} \end{aligned} \tag{2.2}$$

by Chernick (1981) and Chernick and Davis (1982), irrespectively of the behaviour of the sampling sequence (c_n). Therefore, the only non-trivial case as far as the limit in (2.1) is concerned is the case $0<y<x$.

We state and discuss below the limiting results for the positively correlated uniform $AR(1)$ processes and negatively correlated uniform $AR(1)$ processes separately. Before doing so, we would like to draw the attention of the reader to the difference in the mechanism that makes the extremes cluster in the positively correlated case and in the negatively correlated case, and we do it in an informal manner. In the positively correlated case, one large value of the process will be followed by large values as long as the subsequent noise variables keep taking the value $(r-1)/r$. In the negatively correlated case a large value of the process cannot be immediately followed by another large value, but the value two units of time away can be also large as long as the pair of the two subsequent noise variables takes the value $(1/r, 1)$. This importance of the parity in the negatively correlated case will be visible both in the statement of the results and in their proofs.

A. *Positively correlated uniform $AR(1)$ processes.* For a sampling sequence (c_n) we denote for $n,j=1,2,\dots$ the empirical frequencies of consecutive zeroes,

$$f_{n,j} = \sum_{i=1}^{n-j+1} \mathbf{1}\big(c_i = c_{i+1} = \ldots = c_{i+j-1} = 0\big) \tag{2.3}$$

if $j\leqslant n$ and $f_{n,j}=0$ if $j>n$. We will use the notation

$$f_j = \lim_{n\to\infty} \frac{f_{n,j}}{n}, \quad j\in\{1,2,\ldots\}, \tag{2.4}$$

if the limit exists. Note that $f_1 = 1-p$, where p is defined in (1.2), and we always assume existence of this limit. The following theorem is our main result for positively correlated uniform $AR(1)$ processes.

Theorem 2.1. *Let $\mathbf{X}$ be a positively correlated uniform $AR(1)$ process.*

(a) *If the limiting frequencies f_j in (2.4) exist for each $j\in\{1,2,\ldots\}$, then the sequence $\big(n(1-\widetilde{M}_n), n(1-M_n)\big)$, $n=1,2,\ldots$ converges weakly as $n\to\infty$.*

(b) *Let j be a positive integer, and suppose that the limit f_k in (2.4) exists for all $k\in\{1,\ldots,j\}$. Then the limit $G(x,y)$ in (2.1) exists for all (x,y) in the range*

$0 < xr^{-j} \leqslant y < xr^{-(j-1)}$, and is given by

$$G(x,y) = \exp\left\{-(r-1)x \sum_{k=0}^{j-1} \frac{f_k - f_{k+1}}{r^{k+1}} - (r-1)y\, \frac{f_j}{r}\right\}, \tag{2.5}$$

with the convention $f_0 = 1$.

Remark 2.2. An immediate conclusion from Theorem 2.1 and the pointwise ergodic theorem is that, if the sampling sequence (c_n) is a realization of a stationary $0-1$-valued process, then for almost every such realization, the sequence $\big(n(1-\widetilde{M}_n), n(1-M_n)\big)$, $n = 1, 2, \ldots$ converges weakly. In particular, if the underlying stationary process is an i.i.d. Bernoulli sequence with probability p of observing 1, then for almost every realization we have $f_j = (1-p)^j$, $j = 1, 2, \ldots$, and the expression (2.5) for the limiting distribution reduces to

$$G(x,y) = \exp\left\{-\frac{(r-1)p}{r+p-1}\big(1-(1-p)^j r^{-j}\big)x - \frac{(r-1)}{r}(1-p)^j y\right\}$$

for (x, y) in the range $0 < xr^{-j} \leqslant y < xr^{-(j-1)}$, $j = 1, 2, \ldots$.

Remark 2.3. Suppose that the sampling sequence (c_n) is periodic, with period $k \geqslant 1$. This sequence is one of the k possible realisations of the stationary process, consisting of taking the sequence (c_n) and erasing a random number N of its initial entries, the random number N having a discrete uniform distribution between 0 and $k-1$. Since each realization has a positive probability, Remark 2.2 applies, and the sequence $\big(n(1-\widetilde{M}_n), n(1-M_n)\big)$, $n = 1, 2, \ldots$ converges weakly. If l is the largest number of consecutive zeroes within a period, we have $f_j > 0$ for $1 \leqslant j \leqslant l$, and $f_j = 0$ for $j > l$. We conclude that the expression (2.5) for the limiting distribution G in the range $0 < xr^{-(l+1)} \leqslant y < xr^{-l}$ remains valid in the entire range $0 < y < xr^{-l}$. The results of Mladenović (2009) and Mladenović and Živadinović (2015) treat such periodic sampling sequences.

Remark 2.4. Instead of considering the joint distribution of the maxima of all observations and of the registered observations, one can consider the joint distribution of the maxima of the registered and of the nonregistered observations. Let

$$\widehat{M}_n = \max_{1 \leqslant i \leqslant n:\, c_i = 0} X_i$$

be the partial maximum of the nonregistered observations. Then

$$P\left(\widetilde{M}_n \leqslant 1 - \frac{x}{n},\, \widehat{M}_n \leqslant 1 - \frac{y}{n}\right) = \begin{cases} P\left(\widetilde{M}_n \leqslant 1 - \frac{x}{n},\, M_n \leqslant 1 - \frac{y}{n}\right) & \text{if } 0 < y \leqslant x \\ P\left(M_n \leqslant 1 - \frac{x}{n},\, \widehat{M}_n \leqslant 1 - \frac{y}{n}\right) & \text{if } 0 < x \leqslant y, \end{cases}$$

and so the sequence $\big(n(1-\widetilde{M}_n), n(1-\widehat{M}_n)\big)$ converges weakly if and only if both sequences $\big(n(1-\widetilde{M}_n), n(1-M_n)\big)$, $n = 1, 2, \ldots$, and $\big(n(1-\widehat{M}_n), n(1-M_n)\big)$, $n = 1, 2, \ldots$, converge weakly, sufficient conditions for which is the existence of the asymptotic frequency of any number of consecutive zeroes and existence of the asymptotic frequency of any number of consecutive ones. Suppose that the asymptotic frequency of ones $p \in (0, 1)$, so that the marginal limits of both random variables $\big(n(1-\widehat{M}_n)\big)$ and $\big(n(1-\widetilde{M}_n)\big)$ are nondegenerate. When are the maxima of the registered observations and of the nonregistered observations asymptotically independent? Theorem 2.1 says that a necessary and sufficient condition for such asymptotic independence is $f_k = f_1$ for all $k \geqslant 1$. That is, asymptotically, zeroes arrive in arbitrarily long groups and then, automatically, ones arrive in arbitrarily long groups as well. Such sampling sequences (c_n) exist. An example is the following sequence. Let $k_j = [j(1-p)/p]$ for $j = 1, 2, \ldots$, and construct a sampling sequence by alternating j consecutive ones with k_j consecutive zeroes, $j = 1, 2, \ldots$. Clearly, $f_1 = 1 - p$ in this case.

B. *Negatively correlated uniform $AR(1)$ processes.* In this case it is natural to consider, for a sampling sequence (c_n), the empirical frequencies of zeroes at consecutive positions of the same parity, so we define

$$\tilde{f}_{n,j} = \sum_{i=1}^{n-2j+2} \mathbf{1}\{c_i = c_{i+2} = \cdots = c_{i+2j-2} = 0\}$$

if $j \leqslant n/2$, and $\tilde{f}_{n,j} = 0$ if $j > n/2$. We will use the notation

$$\tilde{f}_j = \lim_{n\to\infty} \frac{\tilde{f}_{n,j}}{n}, \; j \in \{1, 2, \ldots, \} \tag{2.6}$$

if the limit exists. Note that $\tilde{f}_{n,1} = f_{n,1}$ defined by (2.3), and as before we assume that the limit $p = 1 - \tilde{f}_1 = 1 - f_1$ in (1.2) exists. The following theorem is our main result for negatively correlated uniform $AR(1)$ processes.

Theorem 2.5. *Let $\mathbf{X}$ be a negatively correlated uniform $AR(1)$ process.*

(a) *If the limiting frequencies $\tilde{f}_j$ in* (2.6) *exist for each $j \in \{1, 2, \ldots\}$, then he sequence $\big(n(1 - \widetilde{M}_n), n(1 - M_n)\big)$, $n = 1, 2, \ldots$ converges weakly as $n \to \infty$.*

(b) *Let j be a positive integer, and suppose that the limit $\tilde{f}_k$ in* (2.6) *exists for all $k \in \{1, \ldots, j\}$. Then the limit $G(x, y)$ in* (2.1) *exists for all (x, y) in the range $0 < xr^{-2j} \leqslant y < xr^{-(2j-2)}$, and is given by*

$$G(x, y) = \exp\left\{ -(r^2 - 1)x \sum_{k=0}^{j-1} \frac{\tilde{f}_k - \tilde{f}_{k+1}}{r^{2k+2}} - (r^2 - 1)y \, \frac{\tilde{f}_j}{r^2} \right\}, \tag{2.7}$$

with the convention $\tilde{f}_0 = 1$.

The remarks on Theorem 2.1 that appear above have obvious counterparts in the negatively correlated case. We only address them briefly. If the sampling sequence (c_n) is a realization of a stationary $0 - 1$-valued process, then for almost every realization, the sequence $\big(n(1 - \widetilde{M}_n), n(1 - M_n)\big)$, $n = 1, 2, \ldots$ still converges weakly, and in the special case of an i.i.d. Bernoulli sequence the limiting distribution of the process is

$$G(x, y) = \exp\left\{ -\frac{(r^2 - 1)p}{r^2 + p - 1} \big(1 - (1-p)^j r^{-2j}\big) x - \frac{(r^2 - 1)}{r^2} (1-p)^j y \right\}$$

for (x, y) in the range $0 < xr^{-2j} \leqslant y < xr^{-2(j-1)}$, $j = 1, 2, \ldots$. Further, we always have weak convergence of the sequence $\big(n(1 - \widetilde{M}_n), n(1 - M_n)\big)$ in the case of a periodic sampling scheme, and the requirement for the asymptotic independence of the maxima of the registered observations and of the nonregistered observations is $\tilde{f}_k = \tilde{f}_1$ for all $k \geqslant 1$.

3. Proofs

The argument we will use depends, as do most of the arguments in related statements, on a version of the $D(u_n)$ condition. Since we have to deal with two types of observations, registered observations and nonregistered observations, and since the corresponding arguments in the multivariate cumulative distribution functions take, in general, different values, we need a two-sequence version of the $D(u_n)$ condition. This condition, the so-called $D(u_n, v_n)$ condition (with $u_n = 1 - x/n$, $v_n = 1 - y/n$, $x, y > 0$ in our case) was introduced by Davis (1979), and Chernick and Davis (1982) explained that this condition holds for both positively and negatively correlated uniform $AR(1)$ processes. This implies the following fact. For a fixed positive integer j, let $x, y > 0$ be either in the range $0 < xr^{-j} \leqslant y < xr^{-(j-1)}$ or in the range $0 < xr^{-2j} \leqslant y < xr^{-(2j-2)}$, depending on whether the process is positively correlated or negatively correlated. In the positively correlated case assume that the limit f_k in (2.4) exists for all $k \in \{1, \ldots, j\}$, and in the negatively correlated case assume that the limit $\tilde{f}_k$ in (2.6) exists for all $k \in \{1, \ldots, j\}$. Let (m_n) be a sequence of positive integers such that $m_n \to \infty$, $m_n/n \to 0$ as $n \to \infty$. If the limit

$$\lim_{n\to\infty} \frac{n}{m_n} \left(1 - P\Big(\widetilde{M}_{m_n} \leqslant 1 - \frac{x}{n},\ M_{m_n} \leqslant 1 - \frac{y}{n}\Big)\right) = H(x, y) \tag{3.1}$$

exists, then the limit $G(x, y)$ in (2.1) exists, and

$$G(x, y) = \exp\big\{-H(x, y)\big\}\,. \tag{3.2}$$

We explain why this is true in Section 4.

Proof of Theorem 2.1. Since part (a) of the theorem is an obvious consequence of part (b), we will prove part (b) of the theorem. Throughout the proof we will use the notation

$$z_i = \begin{cases} x, & \text{if } c_i = 1, \\ y, & \text{if } c_i = 0. \end{cases} \tag{3.3}$$

For a positive integer m we denote

$$\begin{aligned} a_m &= P\left(\widetilde{M}_m \leqslant 1 - x/n,\ M_m \leqslant 1 - y/n\right) \\ &= P\left(X_1 \leqslant 1 - z_1/n,\ X_2 \leqslant 1 - z_2/n, \ldots, X_m \leqslant 1 - z_m/n\right)\,. \end{aligned} \tag{3.4}$$

We will let $m = m_n \to \infty$ sufficiently slowly with n so that $r^m/n \to 0$. Note that

$$X_i = r^{-(i-1)} X_1 + \sum_{j=2}^{i} r^{-(i-j)} \varepsilon_j, \quad i = 2, 3, \ldots, m\,.$$

This implies the following fact that we will use repeatedly in the subsequent calculations:

- The inequality $X_i \leqslant 1 - z_i/n$ holds if and only if $X_1 \leqslant r^{i-1} - \sum\limits_{j=2}^{i} r^{j-1}\varepsilon_j - r^{i-1} z_i/n$.

Writing $\varepsilon_j = k_j/r$ with $k_j \in \{0, 1, \ldots, r-1\}$, this condition translates into

$$X_1 \leqslant r^{i-1} - \sum_{j=2}^{i} r^{j-2} k_j - r^{i-1} z_i/n\,. \tag{3.5}$$

We will often use the following elementary fact about the right hand side of (3.5):

$$r^{i-1}-\sum_{j=2}^{i} r^{j-2}k_j \begin{cases} =1, & \text{if } k_j=r-1 \text{ for all } j=2,3,\dots,i,\\ \geqslant 2, & \text{in all other cases.}\end{cases} \tag{3.6}$$

Using the formulation in (3.5), the probability in (3.4) can be written in the form

$$\begin{aligned} a_m &= \sum_{k_2=0}^{r-1}\sum_{k_3=0}^{r-1}\cdots\sum_{k_m=0}^{r-1} r^{-(m-1)}P\biggl(\bigcap_{i=1}^{m}\Bigl\{X_1\leqslant r^{i-1}-\sum_{j=2}^{i} r^{j-2}k_j - r^{i-1}z_i/n\Bigr\}\biggr)\\ &= S_1+S_2+\cdots+S_m, \end{aligned} \tag{3.7}$$

where we have decomposed the full $(m-1)$-tuple sum on the first line by grouping its terms into sums S_l, $1\leqslant l\leqslant m$ according to the following rules:

- The sum S_m consists of the single term $k_2=\cdots=k_m=r-1$.
- For $2\leqslant l\leqslant m-1$, the sum S_l runs over all $k_2,\dots,k_m\in\{0,1,\dots,r-1\}$ such that $k_2=\cdots=k_l=r-1$, but $k_{l+1}\neq r-1$.
- The sum S_1 runs over all $k_2,\dots,k_m\in\{0,1,\dots,r-1\}$ such that $k_2\neq r-1$.

It is easy to compute the number of terms in each sum. We use these numbers as a part of the calculations below. Recall that according to our constraints on the rate of growth of m we have $r^{i-1}z_i/n\in(0,1)$ for all n large enough. For all such n we have

$$\begin{aligned} S_m &= r^{-(m-1)}P\biggl(\bigcap_{i=1}^{m}\Bigl\{X_1\leqslant 1-r^{i-1}z_i/n\Bigr\}\biggr)\\ &= r^{-(m-1)}\biggl(1-\frac{\max_{1\leqslant i\leqslant m} r^{i-1}z_i}{n}\biggr), \end{aligned} \tag{3.8}$$

while for $1\leqslant l\leqslant m-1$, we have

$$\begin{aligned} S_l &= \bigl[(r-1)r^{m-l-1}\bigr]r^{-(m-1)}P\biggl(\bigcap_{i=1}^{l}\Bigl\{X_1\leqslant 1-r^{i-1}z_i/n\Bigr\}\biggr)\\ &= \frac{r-1}{r^l}\biggl(1-\frac{\max_{1\leqslant i\leqslant l} r^{i-1}z_i}{n}\biggr)\\ &= \frac{r-1}{r^l}-\frac{(r-1)\max_{1\leqslant i\leqslant l} r^{i}z_i}{r^{l+1}n} := A_l-B_l\,. \end{aligned} \tag{3.9}$$

Let $j=1,2,\dots$ and consider a pair (x,y) such that $0<xr^{-j}\leqslant y<xr^{-(j-1)}$. Let $l\in\{1,2,\dots,m-1\}$. The following cases are possible in (3.9).

Case 1 For some $k\in\{0,1,\dots,j-1\}$, $c_{l-k}=1$, while $c_{l-k+1}=\ldots=c_l=0$. In this case

$$\max_{1\leqslant i\leqslant l} r^i z_i = r^{l-k}x\,,$$

so that

$$B_l=\frac{(r-1)x}{r^{k+1}n}\,.$$

Note that this scenario is feasible only if $l-k\geqslant 1$.

Case 2 $c_d=0$ for all integer d such that $\max\{l-j+1,1\}\leqslant d\leqslant l$. In this case

$$\max_{1\leqslant i\leqslant l} r^i z_i = r^l y\,,$$

so that

$$B_l = \frac{(r-1)y}{rn}\,.$$

We conclude that each occurrence of the pattern $10\ldots0$ (one followed by k zeroes), $k \in \{0, 1, \ldots, j-1\}$, in the sequence $c_1, \ldots, c_{m-1}$ contributes exactly

$$\frac{(r-1)x}{r^{k+1}n}$$

to the sum $B_1 + \ldots + B_{m-1}$, while each occurrence of the pattern $0\ldots0$ (j zeroes) contributes exactly

$$\frac{(r-1)y}{rn}$$

to that sum. It is elementary that the number of times the pattern $10\ldots0$ (one followed by k zeroes) occurs in the sequence $c_1, \ldots, c_{m-1}$ is equal to $f_{m,k} - f_{m,k+1} - \delta$, with $\delta \in \{0, 1\}$ ($\delta = 1$ if the number of zeroes at the initial positions of the sequence (c_n) is at least k) and the convention that $f_{m,0} = m$. Therefore,

$$\begin{aligned} S_1 + \ldots + S_{m-1} =& \frac{1}{r} - \frac{1}{r^m} \\ & - \frac{(r-1)x}{n} \sum_{k=0}^{j-1} \frac{f_{m,k} - f_{m,k+1}}{r^{k+1}} - \frac{(r-1)y}{rn} f_{m,j} + O(1/n)\,, \end{aligned}$$

where the $O(1/n)$ term comes both from the δ correction above and from the fact that the sum $B_1 + \ldots + B_{m-1}$ can also contain some additional terms $(r-1)y/(rn)$ due to a possible presence of a string of initial zeroes in the sequence $c_1, \ldots, c_{m-1}$ of the length smaller than j. We conclude by (3.7) that

$$a_m = 1 - \frac{(r-1)x}{n} \sum_{k=0}^{j-1} \frac{f_{m,k} - f_{m,k+1}}{r^{k+1}} - \frac{(r-1)y}{rn} f_{m,j} + O(1/n)\,.$$

Since $m = m_n \to \infty$ as $n \to \infty$, we conclude by (3.4) that

$$\begin{aligned} & \frac{n}{m_n} \Big(1 - P\Big(\widetilde{M}_{m_n} \leqslant 1 - x/n,\, M_{m_n} \leqslant 1 - y/n\Big)\Big) \qquad (3.10) \\ =& (r-1)x \sum_{k=0}^{j-1} \left(\frac{f_{m_n,k}}{m_n} - \frac{f_{m_n,k+1}}{m_n}\right) r^{-(k+1)} + \frac{r-1}{r} y \cdot \frac{f_{m_n,j}}{m_n} + o(1) \\ \to& (r-1)x \sum_{k=0}^{j-1} \frac{f_k - f_{k+1}}{r^{k+1}} + (r-1)y \frac{f_j}{r} \end{aligned}$$

as $n \to \infty$. Therefore, (3.1) holds. An appeal to (3.2) proves (2.5) and, hence, establishes part (b) of the theorem. □

Proof of Theorem 2.5. As in the proof of Theorem 2.1, it is enough to prove part (b). We use once again the notation (3.4), but in this case it is convenient for us to let the sequence $m = m_n$ to consist of even numbers, so for $0 < y < x$ and a positive even integer $2m$ we will consider

$$\begin{aligned} a_{2m} &= P\Big\{\widetilde{M}_{2m} \leqslant 1 - \ x/n, M_{2m} \leqslant 1 - y/n\Big\} \\ &= P\Big\{X_1 \leqslant 1 - z_1/n, X_2 \leqslant 1 - z_2/n, \ldots, X_{2m} \leqslant 1 - z_{2m}/n\Big\}, \qquad (3.11) \end{aligned}$$

where z_i is once again given by (3.3). It is appropriate now to let $m = m_n \to \infty$ sufficiently slowly with n so that $r^{2m}/n \to 0$. The parity of the time variables is important now, so we will use a more detailed representation of the stationary sequence (X_n) than in the proof of Theorem 2.1. Let us write

$$X_{2i-1} = r^{-2i+2}X_1 + \sum_{j=2}^{i} r^{-2i+2j}\varepsilon_{2j-1} - \sum_{j=1}^{i-1} r^{-2i+2j+1}\varepsilon_{2j}, \quad i \geqslant 2,$$
$$X_{2i} = -r^{-2i+1}X_1 - \sum_{j=1}^{i-1} r^{-2i+2j+1}\varepsilon_{2j+1} + \sum_{j=1}^{i} r^{-2i+2j}\varepsilon_{2j}, \quad i \geqslant 1.$$

Certain simple facts follow from the above representations. We will list them separately as they refer to the odd-numbered observations and to the even-numbered observations. We write for each j $\varepsilon_j = k_j/r$ with $k_j \in \{1, \dots, r\}$.

The first simple fact is that for $i \geqslant 2$ the inequality $X_{2i-1} \leqslant 1 - z_{2i-1}/n$ holds if and only if

$$X_1 \leqslant r^{2i-2} + \sum_{j=1}^{i-1} r^{2j-2}k_{2j} - \sum_{j=2}^{i} r^{2j-3}k_{2j-1} - r^{2i-2}z_{2i-1}/n. \tag{3.12}$$

We will use the following elementary fact about the right hand side of (3.12):

$$r^{2i-2} + \sum_{j=1}^{i-1} r^{2j-2}k_{2j} - \sum_{j=2}^{i} r^{2j-3}k_{2j-1}$$
$$\begin{cases} = 1, & \text{if } k_2 = k_4 = \cdots = k_{2i-2} = 1, \ \ k_3 = k_5 = \cdots = k_{2i-3} = r, \\ \geqslant 2, & \text{in all other cases.} \end{cases} \tag{3.13}$$

The second simple fact is that for $i \geqslant 1$ the inequality $X_{2i} \leqslant 1 - z_{2i}/n$ holds if and only if

$$X_1 \geqslant -r^{2i-1} - \sum_{j=1}^{i-1} r^{2j-1}k_{2j+1} + \sum_{j=1}^{i} r^{2j-2}k_{2j} + r^{2i-1}z_{2i}/n. \tag{3.14}$$

Once again, we will use an elementary fact about the right hand side of (3.14):

$$-r^{2i-1} - \sum_{j=1}^{i-1} r^{2j-1}k_{2j+1} + \sum_{j=1}^{i} r^{2j-2}k_{2j}$$
$$\begin{cases} = 0, & \text{if } k_2 = k_4 = \cdots = k_{2i} = r, \ \ k_3 = k_5 = \cdots = k_{2i-1} = 1, \\ \leqslant -1, & \text{in all other cases.} \end{cases} \tag{3.15}$$

We proceed with a decomposition of the probability in (3.11) parallel to the decomposition in (3.7), but now we have to be careful about parity of the time stamp.

We write

$$
\begin{aligned}
a_{2m} = & \sum_{k_2=1}^{r} \sum_{k_3=1}^{r} \cdots \sum_{k_{2m}=1}^{r} r^{-2m+1} P\Bigg(\big\{ X_1 \leqslant 1 - z_1/n \big\} \cap \\
& \bigcap_{i=2}^{m} \Big\{ X_1 \leqslant r^{2i-2} + \sum_{j=1}^{i-1} r^{2j-2} k_{2j} - \sum_{j=2}^{i} r^{2j-3} k_{2j-1} - r^{2i-2} z_{2i-1}/n \Big\} \\
& \bigcap_{i=1}^{m} \Big\{ X_1 \geqslant -r^{2i-1} - \sum_{j=1}^{i-1} r^{2j-1} k_{2j+1} + \sum_{j=1}^{i} r^{2j-2} k_{2j} + r^{2i-1} z_{2i}/n \Big\} \Bigg) \\
= & S_1 + S_2 + \cdots + S_{2m},
\end{aligned} \tag{3.16}
$$

where we have decomposed the full $(2m-1)$-tuple sum on the first line by grouping, once again, its terms into sums S_i, $1 \leqslant i \leqslant 2m$, according to the following rules:

The sum S_{2m} consists of the single term with $k_2 = k_4 = \cdots = k_{2m} = r$ and $k_3 = k_5 = \cdots = k_{2m-1} = 1$. If we select n so large that $z_1/n + r^{j-1} z_j/n \in (0, 1)$ for all $2 \leqslant j \leqslant 2m$ (this choice of n we remain in force for the duration of the proof), then

$$
\begin{aligned}
S_{2m} = & \; r^{-2m+1} P\Big\{ X_1 \leqslant 1 - \frac{z_1}{n}, X_1 \geqslant \max_{1 \leqslant i \leqslant m} \frac{r^{2i-1} z_{2i}}{n} \Big\} \\
= & \; r^{-2m+1} \bigg(1 - \frac{z_1}{n} - \frac{\max_{1 \leqslant i \leqslant m} r^{2i-1} z_{2i}}{n} \bigg).
\end{aligned} \tag{3.17}
$$

The sum S_{2m-1} consists of r terms with $k_2 = k_4 = \cdots = k_{2m-2} = 1$ and $k_3 = k_5 = \cdots = k_{2m-1} = r$, while k_{2m} can take any value in $\{1, 2, \ldots, r\}$. We have

$$
\begin{aligned}
S_{2m-1} = & \; \frac{r}{r^{2m-1}} P\Big\{ X_1 \leqslant 1 - \frac{z_1}{n}, X_1 \leqslant 1 - \max_{2 \leqslant i \leqslant m} \frac{r^{2i-2} z_{2i-1}}{n} \Big\} \\
= & \; r^{-2m+2} \bigg(1 - \frac{\max_{1 \leqslant i \leqslant m} r^{2i-2} z_{2i-1}}{n} \bigg).
\end{aligned} \tag{3.18}
$$

For $1 \leqslant l \leqslant m-1$, the sum S_{2l} runs over all $k_2, k_3, \ldots, k_{2m} \in \{1, 2, \ldots, r\}$ such that $k_2 = \cdots = k_{2l} = r$ and $k_3 = \cdots = k_{2l-1} = 1$, but $(k_{2l+1}, k_{2l+2}) \neq (1, r)$. Clearly, this sum has $(r^2-1) r^{2m-2l-2}$ terms and

$$
\begin{aligned}
S_{2l} = & \; \frac{r^2-1}{r^{2l+1}} \bigg(1 - \frac{z_1}{n} - \max_{1 \leqslant i \leqslant l} \frac{r^{2i-1} z_{2i}}{n} \bigg) \\
= & \; \frac{r^2-1}{r^{2l+1}} \bigg(1 - \frac{z_1}{n} \bigg) - \frac{(r^2-1) \max_{1 \leqslant i \leqslant l} r^{2i} z_{2i}}{r^{2l+2} n} := A_{2l} - B_{2l}.
\end{aligned} \tag{3.19}
$$

For $2 \leqslant l \leqslant m-1$, the sum S_{2l-1} is taken over all $k_2, k_3, \ldots, k_{2m} \in \{1, 2, \ldots, r\}$ such that $k_2 = \cdots = k_{2l-2} = 1$ and $k_3 = \cdots = k_{2l-1} = r$, but $(k_{2l}, k_{2l+1}) \neq (1, r)$. This sum has $(r^2-1) r^{2m-2l-1}$ terms and

$$
\begin{aligned}
S_{2l-1} = & \; \frac{r^2-1}{r^{2l}} \bigg(1 - \max_{1 \leqslant i \leqslant l} \frac{r^{2i-2} z_{2i-1}}{n} \bigg) \\
= & \; \frac{r^2-1}{r^{2l}} - \frac{(r^2-1) \max_{1 \leqslant i \leqslant l} r^{2i-1} z_{2i-1}}{r^{2l+1} n} := A_{2l-1} - B_{2l-1}.
\end{aligned} \tag{3.20}
$$

Finally, the sum S_1 runs over all $k_2, k_3, \dots, k_{2m} \in \{1, 2, \dots, r\}$ such that both $k_2 \neq r$ and $(k_2, k_3) \neq (1, r)$. This sum has $r^{2m-1} - r^{2m-2} - r^{2m-3}$ identical terms, so that

$$S_1 = \big(1 - r^{-1} - r^{-2}\big)\big(1 - z_1/n\big). \tag{3.21}$$

We proceed now in a manner similar to the steps we took to prove Theorem 2.1. Let $j = 1, 2, \dots$ and consider a pair (x, y) such that $0 < xr^{-2j} \leqslant y < xr^{-(2j-2)}$. We consider $l \in \{2, \dots, 2m-2\}$. The following cases are possible in (3.19) and (3.20).

Case 1 If for some $k \in \{0, 1, \dots, j-1\}$, $c_{2l-2k} = 1$, while $c_{2l-2k+2} = \cdots = c_{2l} = 0$, then $\max_{1\leqslant i\leqslant l} r^{2i} z_{2i} = r^{2l-2k} x$ and hence B_{2l} is equal to

$$\frac{(r^2-1)x}{r^{2k+2}n}. \tag{3.22}$$

Similarly, if for some $k \in \{0, 1, \dots, j-1\}$, $c_{2l-2k-1} = 1$, while $c_{2l-2k+1} = \cdots = c_{2l-1} = 0$, then $\max_{1\leqslant i\leqslant l} r^{2i-1} z_{2i-1} = r^{2l-2k-1} x$ and B_{2l-1} is still given by (3.22).

Case 2 If $c_{2d} = 0$ for all even integers $2d$ such that $\max\{2l-2j+2, 2\} \leqslant 2d \leqslant 2l$, then $\max_{1\leqslant i\leqslant l} r^{2i} z_{2i} = r^{2l} y$, and B_{2l} is equal to

$$\frac{(r^2-1)y}{r^2 n}. \tag{3.23}$$

If $c_{2d-1} = 0$ for all odd integers $2d-1$ such that $\max\{2l-2k-1, 1\} \leqslant 2d-1 \leqslant 2l-1$, then $\max_{1\leqslant i\leqslant l} r^{2i-1} z_{2i-1} = r^{2l-1} y$, and B_{2l-1} is still given by (3.23).

Let us denote by $\Pi_k(10\dots0)$ the pattern *one followed by k zeroes on consecutive positions of the same parity*, where $k \in \{0, 1, \dots, j-1\}$. Each occurence of such pattern in the sequence $c_2, \dots, c_{2m-2}$ contributes exactly

$$\frac{(r^2-1)x}{r^{2k+2}n}$$

to the sum $B_2 + \cdots + B_{2m-2}$. Each occurence of the pattern "j zeroes on the consecutive positions of the same parity" contributes exactly

$$\frac{(r^2-1)y}{r^2 n}$$

to that sum. The number of times the pattern $\Pi_k(10\dots0)$ occurs in the sequence $c_1, \dots, c_{2m-2}$ is equal to $\tilde{f}_{2m-2,k} - \tilde{f}_{2m-2,k+1} - \delta$, where $\delta \in \{0, 1, 2\}$ depends on the initial strings of zeroes on even and odd positions in the sequence (c_n), with the convention $\tilde{f}_{2m,0} = 2m$. As in the proof of Theorem 2.1 we conclude that

$$a_{2m} = 1 - \frac{(r^2-1)x}{n} \sum_{k=0}^{j-1} \frac{\tilde{f}_{2m-2,k} - \tilde{f}_{2m-2,k+1}}{r^{2k+2}} - \frac{(r^2-1)y}{r^2 n} \tilde{f}_{2m-2,j} + O(1/n).$$

Since $m = m_n \to \infty$ as $n \to \infty$, we conclude by (2.6) that

$$\begin{aligned}
&\frac{n}{2m}\Big(1 - P\Big(\widetilde{M}_{2m} \leqslant 1 - \frac{x}{n},\ M_{2m} \leqslant 1 - \frac{y}{n}\Big)\Big) \\
&= (r^2-1)x \sum_{k=0}^{j-1} \left(\frac{\tilde{f}_{2m-2,k}}{2m} - \frac{\tilde{f}_{2m-2,k+1}}{2m}\right) r^{-(2k+2)} + (r^2-1)y\,\frac{\tilde{f}_{2m-2,j}}{2mr^2} + o(1) \\
&\to (r^2-1)x \sum_{k=0}^{j-1} \frac{\tilde{f}_k - \tilde{f}_{k+1}}{r^{2k+2}} + (r^2-1)y\,\frac{\tilde{f}_j}{r^2}
\end{aligned}$$

as $n \to \infty$, so (3.1) holds, and the statement of part (b) of the theorem follows from (3.2). □

4. Condition $D(u_n, v_n)$

In this section we prove the claim made at the beginning of Section 3. Specifically, if $x, y > 0$ are in the range $0 < xr^{-j} \leqslant y < xr^{-(j-1)}$ for some positive integer j, then, under the assumption that the limit f_k in (2.4) exists for all $k = 1, \ldots, j$, the statement (3.1) for positively correlated uniform $AR(1)$ processes implies that the limit in (2.1) exists, and is given by (3.2). The argument in the negatively correlated case (under the assumption that the appropriate limits in (2.6) exist) is similar.

We start with the definition of the $D(u_n, v_n)$ condition and a discussion of how it applies to the uniform $AR(1)$ processes.

Definition 4.1. Let (X_n) be a stationary process and (u_n) and (v_n) two sequences of real numbers. *The condition* $D(u_n, v_n)$ is satisfied, if for all sets A_1, A_2, B_1, B_2 of positive integers such that $A_1 \cap A_2 = \emptyset$, $B_1 \cap B_2 = \emptyset$ and

$$b - a \geqslant l \quad \text{for all} \quad a \in A_1 \cup A_2, \ \ b \in B_1 \cup B_2,$$

the following inequality holds

$$\begin{aligned}
&\Bigg| P\Big(\bigcap_{j \in A_1 \cup B_1} \{X_j \leqslant u_n\} \cap \bigcap_{j \in A_2 \cup B_2} \{X_j \leqslant v_n\}\Big) - \\
&- P\Big(\bigcap_{j \in A_1} \{X_j \leqslant u_n\} \cap \bigcap_{j \in A_2} \{X_j \leqslant v_n\}\Big) \cdot P\Big(\bigcap_{j \in B_1} \{X_j \leqslant u_n\} \cap \bigcap_{j \in B_2} \{X_j \leqslant v_n\}\Big)\Bigg| \\
&\leqslant \alpha_{n,l},
\end{aligned} \tag{4.1}$$

with $\alpha_{n,l_n} \to 0$ as $n \to \infty$ for some $l_n = o(n)$.

We claim that the positively correlated uniform $AR(1)$ processes satisfy the bound (4.1) with $u_n = 1 - y/n$, $v_n = 1 - x/n$, where $x > 0$, $y > 0$. Moreover, we can choose $\alpha_{n,l}$ to be independent of $l \geq 1$, but depending instead on the cardinality of $A_1 \cup A_2$. Specifically, (4.1) holds with the right hand side given by

$$\alpha_{n,m} = \frac{\max(x,y)}{n}\,\frac{\log(m \max(x,y))}{\log r} + \frac{1}{n}\,\frac{r}{1 - 1/r}, \tag{4.2}$$

where $m = card(A_1 \cup A_2)$.

In order to establish the above fact, denote

$$\begin{aligned}
&\widetilde{A}_1(u_n) = \bigcap_{j\in A_1}\{X_j \leqslant u_n\}, \quad \widetilde{A}_2(v_n) = \bigcap_{j\in A_2}\{X_j \leqslant v_n\},\\
&\widetilde{B}_1(u_n) = \bigcap_{j\in B_1}\{X_j \leqslant u_n\}, \quad \widetilde{B}_2(v_n) = \bigcap_{j\in B_2}\{X_j \leqslant v_n\},\\
&\Delta_n(A_1,A_2,B_1,B_2) = P\left(\widetilde{A}_1(u_n)\widetilde{A}_2(v_n)\widetilde{B}_1(u_n)\widetilde{B}_2(v_n)\right)\\
&\qquad\qquad -P\left(\widetilde{A}_1(u_n)\widetilde{A}_2(v_n)\right)\cdot P\left(\widetilde{B}_1(u_n)\widetilde{B}_2(v_n)\right)
\end{aligned}$$

Since the process (X_n) is, obviously, associated, it follows that

$$\begin{aligned}
&0 \leqslant \Delta_n(A_1,A_2,B_1,B_2)\\
&= P\left(\widetilde{A}_1(u_n)\widetilde{A}_2(v_n)\right)\left\{P\left(\widetilde{B}_1(u_n)\widetilde{B}_2(v_n)|\widetilde{A}_1(u_n)\widetilde{A}_2(v_n)\right) - P\left(\widetilde{B}_1(u_n)\widetilde{B}_2(v_n)\right)\right\}\\
&:= P\left(\widetilde{A}_1(u_n)\widetilde{A}_2(v_n)\right)(P_1 - P_2).
\end{aligned}$$

Let $i_* = \max(A_1 \cup A_2)$. To bound the difference $P_1 - P_2$ we will use a coupling argument, and we start by outlining its general structure. Let $X_0, \hat{X}_0, \varepsilon_i, i = 1, 2, \ldots$ be random variables defined on some probability space, satisfying the following conditions. Both X_0 and $\hat{X}_0$ take values in $(0,1)$, $\hat{X}_0$ has the standard uniform distribution, and $\hat{X}_0 - X_0 \leq \theta$ a.s. for some nonrandom $\theta \in (0,1)$. Further, $(X_0, \hat{X}_0)$ are independent of an i.i.d sequence (ε_i) of random variables taking values $\{0, 1/r, \ldots, (r-1)/r\}$ with equal probabilities. Define

$$X_i = r^{-i}X_0 + \sum_{j=1}^{i} r^{-(i-j)}\varepsilon_j, \; i = 1, 2, \ldots$$

$$\hat{X}_i = r^{-i}\hat{X}_0 + \sum_{j=1}^{i} r^{-(i-j)}\varepsilon_j, \; i = 1, 2, \ldots.$$

Let C and D be two disjoint finite subsets of $\{l, l+1, \ldots\}$. Then

$$\begin{aligned}
&P\big(X_i \leqslant u_n, i\in C,\, X_i \leqslant v_n, i \in D\big)\\
\leqslant &P\big(\hat{X}_i - r^{-i}\theta \leqslant u_n, i\in C,\, \hat{X}_i - r^{-i}\theta \leqslant v_n, i \in D\big)\\
\leqslant &P\big(\hat{X}_i \leqslant u_n, i\in C,\, \hat{X}_i \leqslant v_n, i \in D\big)\\
+&\sum_{i\in C} P(u_n < \hat{X}_i \leqslant u_n + r^{-i}\theta) + \sum_{i\in D} P(v_n < \hat{X}_i \leqslant v_n + r^{-i}\theta)\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&P\big(X_i \leqslant u_n, i\in C,\, X_i \leqslant v_n, i \in D\big) - P\big(\hat{X}_i \leqslant u_n, i\in C,\, \hat{X}_i \leqslant v_n, i \in D\big) \quad (4.3)\\
\leq &\sum_{i=l}^{\infty} \min\big(r^{-i}\theta,\, \max(x,y)/n\big) = \sum_{l\leqslant i\leqslant \log(n\theta)/\log r} \cdot + \sum_{i>\log(n\theta)/\log r} \cdot\\
\leq &\frac{\max(x,y)}{n}\frac{\log(n\theta)}{\log r} + \frac{1}{n}\frac{r}{1-1/r}.
\end{aligned}$$

We now use (4.3) to estimate the difference $P_1 - P_2$. To this end we need to couple two random variables, X_{i_*} with its unconditional standard uniform distribution, and the same X_{i_*} with its conditional law given $X_i \le u_n, i \in A_1$ and $X_i \le v_n, i \in A_2$. In the coupling the latter random variable is X_0 in the computation leading to (4.3), and the former random variable is $\hat{X}_0$. We need to couple these random variables in such a way that the resulting difference θ is small.

Note that the conditional law of X_{i_*} given $X_i \le u_n, i \in A_1$ and $X_i \le v_n, i \in A_2$ is uniform on some subset of $(0,1)$, consisting of some finite collection of intervals. Each condition of the type $X_i \le u_n, i \in A_1$ removes some subintervals of $(0,1)$ of total length y/n from the support of the conditional law of X_{i_*}, while each condition of the type $X_i \le v_n, i \in A_2$ removes some subintervals of $(0,1)$ of total length x/n from the same support. If $m = card(A_1 \cup A_2)$, then the total number of conditions is m, and the conditional law of X_{i_*} given $X_i \le u_n, i \in A_1$ and $X_i \le v_n, i \in A_2$ is uniform on a finite union of subintervals of $(0,1)$ of total length at least $1 - (m/n)\max(x,y)$. According to Lemma 4.2 below, we can achieve coupling of X_0 and $\hat{X}_0$ with $\theta = (m/n)\max(x,y)$. Therefore, by (4.3),

$$P_1 - P_2 \le \frac{\max(x,y)}{n}\frac{\log(m\max(x,y))}{\log r} + \frac{1}{n}\frac{r}{1-1/r}$$

and, hence, (4.1) holds with the bound in (4.2).

Lemma 4.2. *Let $0 < \gamma < 1$ and let X_0 be a random variable with the uniform distribution on a finite union of disjoint subintervals of $(0,1)$ of total length γ. Then there is a coupling of X_0 with a standard uniform random variable $\hat{X}_0$ such that $\hat{X}_0 - X_0 \leqslant 1-\gamma$ a.s..*

Proof. Let the support of X_0 be the disjoint union of (h_i, h_i+p_i), $i = 1,\dots,k$ with $0 \leqslant h_1 < h_1+p_1 < h_2 < h_2+p_2 < \dots < h_k < h_k+p_k \leqslant 1$ with $p_1+p_2+\dots+p_k = \gamma$. We couple X_0 and $\hat{X}_0$ as follows. Generate X_0. If $X_0 \in (h_i, h_i+p_i)$ for some $i = 1,\dots,k$, we set

$$\hat{X}_0 = X_0/\gamma + (p_1 + \dots + p_{i-1} - h_i)/\gamma.$$

It is elementary to check that $\hat{X}_0$ has the standard uniform distribution. If $X_0 \in (h_i, h_i+p_i)$ for some $i = 1,\dots,k$, we have

$$\begin{aligned}
\hat{X}_0 - X_0 &= (1/\gamma - 1)\,X_0 + (p_1 + \dots + p_{i-1} - h_i)/\gamma \\
&\leqslant (1/\gamma - 1)\,(h_i + p_i) + (p_1 + \dots + p_{i-1} - h_i)/\gamma \\
&= -\,(h_i + p_i) + (p_1 + \dots + p_{i-1})/\gamma \\
&\leqslant -\,(p_1 + \dots + p_{i-1} + p_i) + (p_1 + \dots + p_{i-1} + p_i)/\gamma \\
&= (1/\gamma - 1)\,(p_1 + \dots + p_{i-1} + p_i) \\
&\leqslant (1/\gamma - 1)\,\gamma = 1 - \gamma,
\end{aligned}$$

as required. □

We are now ready to prove that (3.1) implies that the limit $G(x,y)$ in (2.1) exists and (3.2) holds. Let $m = m_n \to \infty$ be a sequence of positive integers such that $m_n/n \to 0$ and (3.1) holds. Choose another sequence of positive integers, $l = l_n \to \infty$ such that $l_n/m_n \to 0$ as $n \to \infty$. Let $k = [n/(m+l)]$. For $0 \leqslant y \leqslant x$ we consider the difference

$$\Delta = P\left\{\widetilde{M}_n \leqslant 1 - x/n,\, M_n \leqslant 1 - y/n\right\} - \left[P\Big(\widetilde{M}_m \leqslant 1 - \ x/n,\, M_m \leqslant 1 - y/n\Big)\right]^k.$$

We will prove that $\Delta \to 0$ as $n \to \infty$. This will clearly imply that the limit in (2.1) exists, and is given by (3.2). Decompose $\mathbb{N}_n = \{1, 2, \ldots, n\} = (I_1 \cup J_1) \cup (I_2 \cup J_2) \cup \cdots \cup (I_k \cup J_k) \cup I_{k+1}$ with

$$\begin{aligned}
I_i &= \{(m+l)(i-1)+1, \ldots, (m+l)(i-1)+m\}, \quad 1 \leqslant i \leqslant k, \\
J_i &= \{mi + l(i-1)+1, \ldots, mi + l(i-1)+l\}, \quad 1 \leqslant i \leqslant k, \\
I_{k+1} &= \{(m+l)k+1, \ldots, n\}.
\end{aligned}$$

For $1 \leqslant i \leqslant k$ we set

$$\begin{aligned}
\widetilde{M}(I_i) &= \max\{X_j \,|\, j \in I_i, c_j = 1\}, \\
M(I_i) &= \max\{X_j \,|\, j \in I_i\}, \\
A_i &= \left\{\widetilde{M}(I_i) \leqslant 1 - x/n, M(I_i) \leqslant 1 - y/n\right\}.
\end{aligned}$$

Using the notation $A_1A_2\ldots A_k$ for the intersection of k events, it is clear that $|\Delta|$ does not exceed

$$\begin{aligned}
&\left|P\left\{\widetilde{M}_n \leqslant 1 - x/n,\, M_n \leqslant 1 - y/n\right\} - P\left\{\widetilde{M}_{(m+l)k} \leqslant 1 - x/n,\, M_{(m+l)k} \leqslant 1 - y/n\right\}\right| \\
&\quad + \left|P\left\{\widetilde{M}_{(m+l)k} \leqslant 1 - x/n,\, M_{(m+l)k} \leqslant 1 - y/n\right\} - P(A_1A_2\ldots A_k)\right| \\
&\quad + \left|P(A_1A_2\ldots A_k) - \big[P(A_1)\big]^k\right| \\
&:= \Delta_1 + \Delta_2 + \Delta_3.
\end{aligned}$$

Since $1 - x/n \leqslant 1 - y/n$,

$$\begin{aligned}
&\left\{\widetilde{M}_{(m+l)k} \leqslant 1 - x/n,\, M_{(m+l)k} \leqslant 1 - y/n\right\} \setminus \left\{\widetilde{M}_n \leqslant 1 - x/n,\, M_n \leqslant 1 - y/n\right\} \\
&\qquad\qquad \subset \bigcup_{j=(m+l)k+1}^{n} \{X_j > 1 - x/n\},
\end{aligned}$$

and, hence,

$$\Delta_1 \leqslant (m+l)P\big\{X_1 > 1 - x/n\big\} = (m+l)x/n \to 0 \quad \text{as } n \to \infty.$$

Similarly,

$$\Delta_2 \leqslant lk\, P\big\{X_1 > 1 - x/n\big\} = lkx/n \to 0 \quad \text{as } n \to \infty.$$

Further, we can write

$$\begin{aligned}
\Delta_3 \quad &\leqslant \quad |P(A_1A_2\ldots A_k) - P(A_1)P(A_2)\ldots P(A_k)| \\
&\qquad + \left|P(A_1)P(A_2)\ldots P(A_k) - \big[P(A_1)\big]^k\right| \\
&:= \quad \Delta_3' + \Delta_3''.
\end{aligned}$$

Since each set I_i contains m numbers, by a repeated application of the $D(u_n, v_n)$ bound (4.1) with (4.2), we obtain

$$\Delta_3' \leqslant k\alpha_{n,m} \leqslant \frac{\max(x,y)}{m}\frac{\log(m\max(x,y))}{\log r} + \frac{1}{m}\frac{r}{1 - 1/r} \to 0$$

since $m \to \infty$ as $n \to \infty$, so it remains to consider Δ_3''.

For $1 \leqslant i \leqslant k$ let

$$f_{m,j}^{(i)} = \sum_{i=(m+l)(i-1)+1}^{(m+l)(i-1)+m-j+1} \mathbf{1}(c_i = c_{i+1} = \cdots = c_{i+j-1} = 0)$$

be the number of times the pattern $00\ldots 0$ (j consecutive zeros) appears in the stretch of the sequence $(c_n,\, n \in I_i)$. Since the limit in (2.4) is assumed to exist, we know that $f_{m,j}^{(1)}/m = f_{m,j}/m \to f_j$ as $m \to \infty$ and, similarly, for $i \geqslant 2$,

$$\begin{aligned}\frac{f_{m,j}^{(i)}}{m} &= \frac{f_{(m+l)(i-1)+m,j}}{(m+l)(i-1)+m} \cdot \frac{(m+l)(i-1)+m}{m} \\ &\qquad - \frac{f_{(m+l)(i-1),j}}{(m+l)(i-1)} \cdot \frac{(m+l)(i-1)}{m} \\ &\to f_j \cdot i - f_j \cdot (i-1) = f_j.\end{aligned}$$

as $m \to \infty$. In the proof of Theorem 2.1 we showed that this impies that

$$\begin{aligned}P(A_i) &= 1 - \frac{(r-1)x}{n} \sum_{d=0}^{j-1} \frac{f_{m,d}^{(i)} - f_{m,d+1}^{(i)}}{r^{d+1}} - \frac{(r-1)y}{rn} f_{m,j}^{(i)} + O(1/n) \\ &= \exp\left\{ -\left(\frac{(r-1)x}{n} \sum_{d=0}^{j-1} \frac{f_{m,d}^{(i)} - f_{m,d+1}^{(i)}}{r^{d+1}} - \frac{(r-1)y}{rn} f_{m,j}^{(i)} \right) \right\} + O(1/n)\,,\end{aligned}$$

$i = 1, 2, \ldots$. Consequently,

$$\begin{aligned}&P(A_1)P(A_2)\ldots P(A_k) \\ &= \exp\left\{ -\sum_{i=1}^{k} \left(\frac{(r-1)x}{n} \sum_{d=0}^{j-1} \frac{f_{m,d}^{(i)} - f_{m,d+1}^{(i)}}{r^{d+1}} - \frac{(r-1)y}{rn} f_{m,j}^{(i)} \right) \right\} + O(k/n) \\ &= \exp\left\{ -\left(\frac{(r-1)x}{n} \sum_{d=0}^{j-1} \frac{f_{n,d} - f_{n,d+1}}{r^{d+1}} - \frac{(r-1)y}{rn} f_{n,j} \right) + O(kl/n) \right\} + O(k/n) \\ &\to \exp\bigl\{-H(x,y)\bigr\}\end{aligned}$$

as $n \to \infty$. Since we also have

$$\bigl[P(A_1)\bigr]^k \to e^{-H(x,y)}$$

as $n \to \infty$, we have proved that $\Delta_3'' \to 0$.

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