PRELIMINARY REPORT
Generalized spherical and related multivariate distributions

John P. Nolan,* American University

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1 Introduction

Here are a collection of ideas for modeling multivariate distributions that are not normal. The motivation was to be able to model star-shaped distributions, but the approach allows more general contours. We are developing R code that allows one to work with these classes of distributions: specifying general shapes, computing densities, simulating and fitting data. A deliberate goal in this process is to have methods and programs that work in arbitrary dimension $d \geq 2$.

2 Generalized spherical/homothetic distributions

Fernández et al. (1995) construct multivariate distributions from a contour $\mathcal{C}$ (a simple closed curve/surface in $\mathbb{R}^d$) that is specified by a contour function $c : \mathbb{S} \to [0, \infty)$:

$$\mathcal{C} = \{c(\mathbf{s})\mathbf{s} \, : \, \mathbf{s} \in \mathbb{S}\}.$$

Figure 1 shows an example. Here $\mathbb{S} = \{\mathbf{s} \in \mathbb{R}^d : |\mathbf{x}| = 1\}$ is the unit sphere. We assume that $c(\mathbf{s})$ is a continuous function.

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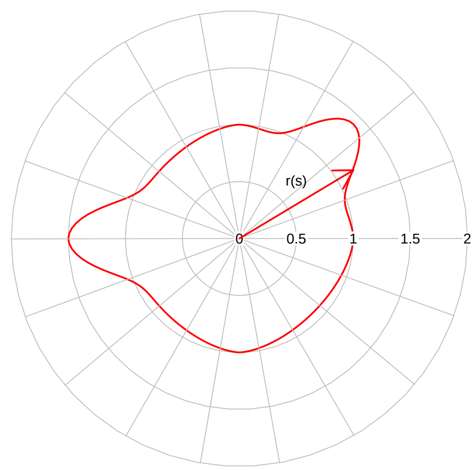

Figure 1: A contour $\mathcal{C}$ and contour function $c(\mathbf{s})$.

Let $g : [0, \infty) \to [0, \infty)$ be a nonnegative function and define

$$f(\mathbf{x}) = \begin{cases} g\left(\frac{|\mathbf{x}|}{c(\mathbf{x}/|\mathbf{x}|)}\right) & |\mathbf{x}| > 0 \\ g(0) & |\mathbf{x}| = 0 \end{cases} \tag{1}$$

Under integrability conditions discussed below, this will give a probability density function on $\mathbb{R}^d$, and the level sets of such a distribution are scalar multiples of $\mathcal{C}$. Such distributions are called *homothetic*. We will call $c(\cdot)$ the *contour function* and $g(\cdot)$ the *radial function* of the distribution.

Our approach differs a bit from Fernández et al. (1995) because we take the contour function $c(\cdot)$ as the basic object, whereas they take $v(\mathbf{x}) := |\mathbf{x}|/c(\mathbf{x}/|\mathbf{x}|)$. By construction, v is homogeneous: $v(a\mathbf{x}) = |a|v(\mathbf{x})$. If $c(\mathbf{s}) = 1$, then $\mathcal{C}$ is the unit sphere and $v(\mathbf{x}) = |\mathbf{x}|$, so the resulting classes of distributions are the spherical/isotropic distributions. If $v(\cdot)$ is convex, then $v(\cdot)$ is a norm and $\mathcal{C}$ is the unit ball in that norm, hence the name v-spherical distributions. When $v(\cdot)$ is not convex, $v(\mathbf{x})$ does not give a norm, so $\mathcal{C}$ is not a unit ball, but we will still call the resulting distributions v-spherical.

Other references: Arnold et al. (2008), Ferreira and Steel (2005), Kamiya et al. (2008), Rattihalli and Basugade (2009), Rattihalli and Patil (2010), and Balkema and Nolde (2010).

2.1 Conditions that guarantee a density function

For (1) to be a proper density, it is required that (see Fernández et al. (1995), eq. (4) and (5))

$$k_{\mathcal{C}}^{-1} := \int_{\mathbb{S}} c^d(\mathbf{s}) d\mathbf{s} \in (0, \infty) \tag{2}$$

and

$$\int_0^\infty r^{d-1} g(r) dr = k_{\mathcal{C}}.$$

To satisfy the integrability condition above, the set $\{\mathbf{s} \in \mathbb{S} : c(\mathbf{s}) = 0\}$ must have surface area measure 0. In the implementation described below, we will assume $c(\cdot)$ is continuous and $\mathbb{S}$ and that $c(\mathbf{s}) \geq c_0 > 0$. This guarantees (2) is finite, though evaluating it may be difficult, especially in higher dimensions.

The polar coordinates version of (2) is $k_{\mathcal{C}}^{-1} := \int_\Theta c^d(t(\boldsymbol{\theta}))|J(\boldsymbol{\theta})|d\boldsymbol{\theta} \in (0, \infty)$, where $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_{d-1}) \in \Theta := (-\pi/2, \pi/2)^{d-2} \times [0, 2\pi)$, $t(\boldsymbol{\theta}) = (\sin\theta_1, \cos\theta_1 \sin\theta_2, \ldots, (\prod_{j=1}^{d-1} \cos\theta_j) \sin\theta_{d-1})$, and $J(\boldsymbol{\theta}) = \prod_{i=1}^{d-2} \cos^{d-1-i}\theta_i$ is the Jacobian of the polar transform.

Given any univariate pdf $h(\cdot)$, the function $g(r) = k_{\mathcal{C}} r^{1-d} h(r)$ is a valid radial function.

2.2 Stochastic representation/simulation

$$\mathbf{X} \stackrel{d}{=} Y \mathbf{Z},$$

where $Y \geq 0$ given by g and $\mathbf{Z}$ is uniformly distributed on the contour $\mathcal{C}$. Can approximate this in $\mathbb{R}^2$ where the contour is specified by a curve. Not clear how to do this in $d \geq 3$.

To sample from the manifold Diaconis et al. (2012), Saucan et al. (2007).

Or, using Balkema and Nolde (2010),

$$\mathbf{X} \stackrel{d}{=} Y^* \mathbf{Z}^*,$$

where $Y^* \geq 0$ given by g and $\mathbf{Z}^*$ is uniformly distributed on the unit disk $\mathcal{D} = \{\mathbf{x} \in \mathbb{R}^d : v(\mathbf{x}) \leq 1\}$. An advantage of this is that it is straightforward, though possibly inefficient, to simulate from $\mathcal{D}$ by generating a uniform vector on a rectangle that contains the ball $\mathcal{D}$ and rejecting if $v(\mathbf{x}) > 1$.

2.3 Specification of the contour function

For modeling purposes, we want a flexible family of functions that can be used in a variety of problems. In particular, we want to be able to model star-shaped contours that arise in munitions fragment patterns. To be able to include the distributions discussed by the authors cited above, we allow contour functions of the form

$$c(\mathbf{s}) = \sum_{j=1}^{N_1} c_j r_j(\mathbf{s}) + \frac{1}{\sum_{j=1}^{N_2} c_j^* r_j^*(\mathbf{s})},$$

where $c_j > 0$, $c_j^* > 0$, and $r_j(\cdot)$ and/or $r^*(\cdot)$ are one of the cases discussed below.

1. $c(\mathbf{s}) = 1$, which makes $\mathcal{C}$ the Euclidean ball.

2. $c(\mathbf{s}) = c(\mathbf{s}|\boldsymbol{\mu}, \theta)$ is a cone with peak 1 at center $\boldsymbol{\mu} \in \mathbb{S}$ and height 0 at the base given by the circle $\{\mathbf{x} \in \mathbb{S} : \boldsymbol{\mu} \cdot \mathbf{x} = \cos\theta\}$. It is assumed that $|\theta| \leq \pi/2$.

3. $c(\mathbf{s}) = c(\mathbf{s}|\boldsymbol{\mu}, \sigma) = \exp(-t(\mathbf{s})^2/(2\sigma^2))$ is a Gaussian bump centered at location $\boldsymbol{\mu} \in \mathbb{S}$ and "standard deviation" $\sigma > 0$. Here $t(\mathbf{s})$ is the distance between $\boldsymbol{\mu}$ and the projection of s $\in \mathbb{S}$ linearly onto the plane tangent to $\mathbb{S}$ at $\boldsymbol{\mu}$.

4. $r^*(\mathbf{s}) = ||\mathbf{s}||_{\ell^p(\mathbb{R}^d)}$, $p > 0$.

5. $r^*(\mathbf{s}) = ||A\mathbf{s}||_{\ell^p(\mathbb{R}^m)}$, $p > 0$, A an $(m \times d)$ matrix. This allows a generalized p-norm. If A is $d \times d$ and orthogonal, then the resulting contour will be a rotation of the standard unit ball in ℓ^p. If A is $d \times d$ and not orthogonal, then the contour will be sheared. If $m > d$, it will give the ℓ^p norm on $\mathbb{R}^m$ of As.

6. $r^*(\mathbf{s}) = (\mathbf{s}^T A\mathbf{s})^{1/2}$, where A is a positive definite $(d \times d)$ matrix. Then the level curves of the distribution are ellipses.

Sums of the first three types allow us to describe star shaped contours, see Figure 2. Inverses of sums of the last two types allow us to consider contours that are familiar unit balls, or generalized unit balls, or sums of such shapes. Combinations allow more general shapes, e.g. Figure 1 is mix of one term of type 4 with $p = 1.6$ and two bumps of type 3, one at angle $\pi/4$ and one at angle π.

We will always assume that $c(\mathbf{s})$ is continuous and bounded away from 0, say $c(\mathbf{s}) \geq c_0 > 0$. The particular forms of types 2 and 3 are to guarantee continuity. The strict positivity is automatic for several of the classes of contour functions

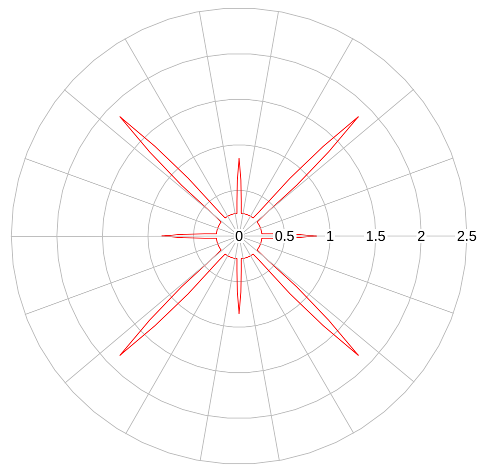

Figure 2: A star shaped region using one term of type 1 and 8 terms of type 2.

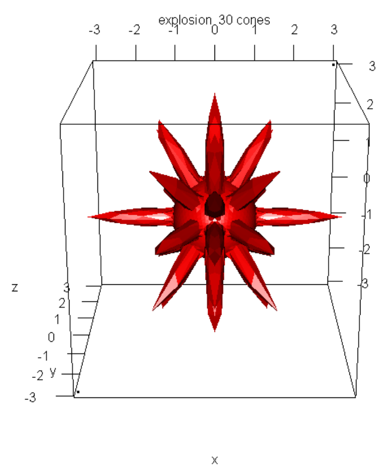

Figure 3: A three dimensional star-shaped region with one term of type 1 and 30 terms of type 2.

considered below, and can always be guaranteed by including an isotropic term with a small coefficient.

Other possible functions can be defined. One possible approach to define more general functions on spheres is to use splines on spheres as described by Ferreira and Steel (2005), see Wahba (1981) and Taijeron et al. (1994).

2.4 Estimation for generalized spherical models

Separate into two parts: (1) determining the radial $g(\cdot)$ and (2) determining the contour $\mathcal{C}$.

Pick $g(\cdot)$.

Pick a class of models, e.g. the number and types of terms allowable in (1). Then one can use (1) to compute the density $f(\mathbf{x})$ and therefore the likelihood. One can numerically maximize the likelihood. For star shaped regions, can allow one term of type 1 (to guarantee $c(\cdot) > 0$) and m terms of type 2 to determine the best location and scale parameters. Can vary m and use AIC to select optimal number of cones.

3 Stable distributions

Multivariate stable with discrete spectral measures. Can simulate using Modarres and Nolan (1994), can compute bivariate pdf using Nolan and Rajput (1995).

The contours of these distributions are star shaped far away from the origin, but round out near the origin. They can be supported on a cone or the whole space.

4 Multivariate extreme value distributions

Multivariate Fréchet with discrete spectral measures. Deheuvels (1983), Einmahl et al. (2011), Fougères et al. (2013). These distributions are supported on the postive orthant, contours are star shaped.

5 Linear combinations of independent terms

Let $\mathbf{Z} = (Z_1, \ldots, Z_m)^T$ be a vector of i.i.d. terms and let A be a $(d \times m)$ matrix of numbers and define the d dimensional r. vector

$$\mathbf{X} = A\mathbf{Z}.$$

In general, $d < m$. When the Z_i are stable, $\mathbf{X}$ is multivariate stable with a discrete spectral measure. These distributions have If each Z_i is infinitely divisible (i.d.), then the vector $\mathbf{Z}$ is i.d. and therefore $\mathbf{X}$ is i.d. This means that we can define an i.d. process $\mathbf{X}_t := A\mathbf{Z}_t$ and try to model the time evolution of the distribution. For example, the Z_i could be i.i.d. $\Gamma(\alpha, 1)$, then $\mathbf{Z}_t$ and $\mathbf{X}_t$ would be gamma processes.

Likely to be unrealistic in practice, as the model continues to propagate out, whereas in real life, fragments would stop. Perhaps scale $a_t\mathbf{X}_t$, with $a_t \to 0$ as $t \to \infty$ to halt propagation.

When other Z_i terms are used, we get a new class of multivariate distributions. When the Z_i terms are heavy tailed, will get star-shaped regions.

6 Questions

1. 2-dim. vs 3-dim (or higher?)
2. Static vs. dynamic model?
3. unobstructed vs. contained/limited dispersion
4. what do we want to do: simulate, compute pdf, cdf, fit data?
5. How much data is there?
6. Goal: provide a few tools vs in depth modeling and analysis?
7. Use of max stable fields for penetrating armor

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