Truncated fractional moments of stable laws

John P. Nolan[1]

American University

Abstract

Expressions are given for the truncated fractional moments EX_+^p of a general stable law. These involve families of special functions that arose out of the study of multivariate stable densities and probabilities. As a particular case, an expression is given for $E(X-a)_+$ when $\alpha > 1$.

Keywords: stable distribution, truncated moments, fractional moments
2000 MSC: 60E07, 60E10

1. Introduction

A univariate stable r.v. Z with index α, skewness β, scale γ, and location δ has characteristic function

$$\phi(u) = \phi(u|\alpha,\beta,\gamma,\delta) = E\exp(iuZ) = \exp(-\gamma\ \ [|u|\ \ + i\beta\eta(u,\alpha)] + iu\delta), \quad (1)$$

where $0 < \alpha \leq 2$, $-1 \leq \beta \leq 1$, $\gamma > 0$, $\delta \in \mathbb{R}$ and

$$\eta(u,\alpha) = \begin{cases} -(\operatorname{sign} u)\tan(\pi\alpha/2)|u| & \alpha \neq 1 \\ (2/\pi)\, u \ln|u| & \alpha = 1. \end{cases}$$

In the notation of Samorodnitsky and Taqqu (1994), this is a $S\ \ (\gamma,\beta,\delta)$ distribution. We will use the notation $X \sim \mathbf{S}(\alpha,\beta,\gamma,\delta;1)$ (the ";1" is used to distinguish between this parameterization and a continuous one used below).

The purpose of this paper is to derive expressions for truncated fractional moments $EX_+^p = E(X\mathbb{1}_{\{X\geq 0\}})^p$ for general stable laws. To do this, define the functions for real x and d

$$g_d(x|\alpha,\beta) \quad = \quad \begin{cases} \displaystyle\int_0^\infty \cos(xr + \beta\eta(r,\alpha))\, r^{d-1} e^{-r}\ \ dr & 0 < d < \infty \\ \displaystyle\int_0^\infty [\cos(xr + \beta\eta(r,\alpha)) - 1]\, r^{d-1}\, e^{-r}\ \ dr & -2\min(1,\alpha) < d \leq 0 \end{cases}$$

Email address: `jpnolan@american.edu` (John P. Nolan)
[1]The author was supported by an agreement with Cornell University, Operations Research & Information Engineering under W911NF-12-1-0385 from the Army Research Development and Engineering Command.

$$\widetilde{g}_d(x|\alpha,\beta) = \begin{cases} \int_0^\infty \sin(xr + \beta\eta(r,\alpha))\, r^{d-1} e^{-r}\ \ dr & -\min(1,\alpha) < d < \infty \\ \int_0^\infty [\sin(xr + \beta\eta(r,\alpha)) - xr]\, r^{d-1} e^{-r}\ \ dr & \alpha > 1,\ -\alpha < d \le -1. \end{cases}$$

The functions $g_d(\cdot|\alpha,\beta)$ and $\widetilde{g}_d(\cdot|\alpha,\beta)$, for integer subscripts $d = 1,2,3,\ldots$ were introduced in Abdul-Hamid and Nolan (1998). (The notation was slightly different there: a factor of $(2\pi)^{-d}$ was included in the definition and $g_{\ ,d}(x,\beta)$ was used instead of $g_d(x|\alpha,\beta)$, while $q_{\ ,1}(x,\beta)$ was used instead of $\widetilde{g}_1(x|\alpha,\beta)$.)

The expressions for EX_+^p will involve the functions $g_{-p}(\cdot|\alpha,\beta)$ and $\widetilde{g}_{-p}(\cdot|\alpha,\beta)$, i.e. negative values fractional values of the subscript d. Before proving that result, we show that the functions $g_d(\cdot|\alpha,\beta)$ and $\widetilde{g}_d(\cdot|\alpha,\beta)$ have multiple uses. For a standardized univariate stable law, Fourier inversion of the characteristic function shows that the d.f. and density are given by

$$\begin{aligned} F(x|\alpha,\beta) - F(0|\alpha,\beta) &= \frac{1}{\pi}\left(\widetilde{g}_0(x|\alpha,\beta) - \widetilde{g}_0(0|\alpha,\beta)\right) \qquad (2) \\ f(x|\alpha,\beta) &= \frac{1}{\pi} g_1(x|\alpha,\beta). \end{aligned}$$

We note that there are explicit formulas for $F(0|\alpha,\beta)$ when $\alpha \neq 1$.

The $g_d(\cdot|\alpha,\beta)$ functions are used in a similar way to give d-dimensional stable densities, see Theorem 1 of Abdul-Hamid and Nolan (1998) (note that there is a sign mistake in that formula when $\alpha = 1$), and Nolan (2017) uses both $g_d(\cdot|\alpha,\beta)$ and $\widetilde{g}_d(\cdot|\alpha,\beta)$ to give an expression for multivariate stable probabilities.

Another use of these functions is in conditional expectation of X_2 given $X_1 = x$ when (X_1, X_2) are jointly stable with zero shift and spectral measure Λ. In general, the conditional expectation is a complicated non-linear function; here it is restated in terms of these functions. If $\alpha > 1$ or ($\alpha \le 1$ and (5.2.4) in Samorodnitsky and Taqqu (1994) holds), then Theorems 5.2.2 and 5.2.3 in Samorodnitsky and Taqqu (1994) show that the conditional expectation exists for x in the support of X_1 and is given by

$$E(X_2|X_1 = x) = \begin{cases} c_1 x + c_2 \left[\dfrac{1 - (x/\gamma_1)\widetilde{g}_1(x/\gamma_1|\alpha,\beta_1)}{g_1(x/\gamma_1|\alpha,\beta_1)/\gamma_1}\right] & \alpha \neq 1 \\ c_0 + c_1 \dfrac{x-\mu_1}{\gamma_1} + c_2 \dfrac{\widetilde{g}_1((x-\mu_1)/\gamma_1 - (2\beta_1/\pi)\ln\gamma_1|1,\beta_1)}{g_1(x/\gamma_1|1,\beta_1)} & \alpha = 1, \beta_1 \neq 0 \\ c_0 + c_1 \dfrac{x-\mu_1}{\gamma_1} & \\ \quad + c_2 \left[\dfrac{(1-\ln\gamma_1) g_1((x-\mu_1)/\gamma_1|1,0) + h_1((x-\mu_1)/\gamma_1|1,0)}{g_1(x/\gamma_1|1,0)}\right] & \alpha = 1, \beta_1 = 0, \end{cases}$$

where β_1 and γ_1 are the skewness and scale parameters of X_1, and the constants and function $h_1(\cdot|1,0)$ are given by

$$c_0 = -\frac{2}{\pi}\int_{\mathbb{S}} s_2 \ln|s_1|\, \Lambda(d\mathbf{s})$$

$$
\begin{aligned}
c_1 &= \begin{cases} \dfrac{\kappa_1 + \beta_1 \tan^2(\pi\alpha/2)\kappa_2}{\gamma_1 \left(1 + \beta_1^2 \tan^2(\pi\alpha/2)\right)} & \alpha \neq 1 \\ \kappa_2/\beta_1 & \alpha = 1, \beta_1 \neq 0 \\ \kappa_1 & \alpha = 1, \beta_1 = 0 \end{cases} \\
c_2 &= \begin{cases} \dfrac{\tan(\pi\alpha/2)(\kappa_2 - \beta_1\kappa_1)}{\gamma_1 \left(1 + \beta_1^2 \tan^2(\pi\alpha/2)\right)} & \alpha \neq 1 \\ (\kappa_2 - \beta_1\kappa_1)/\beta_1 & \alpha = 1, \beta_1 \neq 0 \\ -2\kappa_2/\pi & \alpha = 1, \beta_1 = 0 \end{cases} \\
\kappa_1 &= [X_2, X_1] \quad = \begin{cases} \int_{\mathbb{S}} s_2 s_1^{< \ -1>} \, \Lambda(d\mathbf{s}) & \alpha \neq 1 \\ \int_{\mathbb{S}} s_2 s_1^{<0>} \, \Lambda(d\mathbf{s}) = \int_{\mathbb{S}} s_2 \operatorname{sign}(s_1) \, \Lambda(d\mathbf{s}) & \alpha = 1 \end{cases} \\
\kappa_2 &= \int_{\mathbb{S}} s_2 |s_1| \ ^{-1} \, \Lambda(d\mathbf{s}) \\
\mu_1 &= -\frac{2}{\pi} \int_{\mathbb{S}} s_1 \ln|s_1| \, \Lambda(d\mathbf{s}) \\
h(x|1,0) &= \int_0^\infty \cos(xr)(\log r) e^{-r} dr.
\end{aligned}
$$

In the terms above, $\mathbb{S}$ is the unit circle and $[X_2, X_1]$ is the $\alpha-$covariation. Note that if Λ is symmetric, then $c_0 = \kappa_2 = \beta_1 = \mu_1 = 0$, so $c_2 = 0$ and

$$E(X_2|X_1 = x) = \frac{[X_2, X_1]}{\gamma_1} x$$

is linear.

2. Truncated moments EX_+^p

The main result of this paper is the following expression for the fractional truncated moment of a stable r.v.

Theorem 1. *Let $X \sim \mathbf{S}(\alpha, \beta, \gamma, \delta; 1)$ with any $0 < \alpha < 2$ and any $-1 \leq \beta \leq 1$ and set*

$$\delta^* = \begin{cases} \delta/\gamma & \alpha \neq 1 \\ \delta/\gamma + (2/\pi)\beta \log\gamma & \alpha = 1. \end{cases}$$

For $p < \alpha$, define $m^p(\alpha, \beta, \gamma, \delta) = EX_+^p$.

(a) When $p = 0$,

$$m^0(\alpha, \beta, \gamma, \delta) = P(X > 0) = \frac{1}{2} - \frac{1}{\pi} \widetilde{g}_0(-\delta^*|\alpha, \beta).$$

When $0 < p < \min(1, \alpha)$,

$$
\begin{aligned}
m^p(\alpha, \beta, \gamma, \delta) &= \gamma^p \frac{\Gamma(p+1)}{\pi} \left[\sin\left(\tfrac{\pi p}{2}\right) \left(\frac{\Gamma(1 - p/\alpha)}{p} - g_{-p}(-\delta^*|\alpha, \beta) \right) \right. \\
&\qquad \left. - \cos\left(\tfrac{\pi p}{2}\right) \widetilde{g}_{-p}(-\delta^*|\alpha, \beta) \right].
\end{aligned}
$$

When $p = 1 < \alpha < 2$*,*

$$m^p(\alpha, \beta, \gamma, \delta) = \gamma \left[\frac{\delta^*}{2} + \frac{1}{\pi} \left(\Gamma(1 - 1/\alpha) - g_{-1}(-\delta^* | \alpha, \beta) \right) \right].$$

When $1 < p < \alpha < 2$*,*

$$\begin{aligned} m^p(\alpha, \beta, \gamma, \delta) &= \gamma^p \frac{\Gamma(p+1)}{\pi} \left[\sin\left(\tfrac{\pi p}{2}\right) \left(\frac{\Gamma(1 - p/\alpha)}{p} - g_{-p}(-\delta^* | \alpha, \beta) \right) \right. \\ &\quad \left. + \cos\left(\tfrac{\pi p}{2}\right) \left(\frac{\delta^*}{\alpha} \Gamma((1-p)/\alpha) - \widetilde{g}_{-p}(-\delta^* | \alpha, \beta) \right) \right]. \end{aligned}$$

(b) $EX_-^p = E(-X)_+^p = m^p(\alpha, -\beta, \gamma, -\delta)$.

Proof (a) To simplify calculations, first assume $\gamma = 1$; the adjustment for $\gamma \neq 1$ is discussed below. When $p = 0$, $EX_+^0 = \int_0^\infty 1\, f(x)dx = P(X > 0)$, and (2) and $\widetilde{g}_0(x|\alpha, \beta) \to \pi/2$ as $x \to \infty$ gives the value in terms of $\widetilde{g}_0(\cdot|\alpha, \beta)$. When $0 < p < \min(1, \alpha)$, Corollary 2 of Pinelis (2011) with $k = \ell = 0$ shows

$$EX_+^p = \frac{\Gamma(p+1)}{\pi} \int_0^\infty \Re \frac{\phi(u) - 1}{(iu)^{p+1}} du. \tag{3}$$

First assume $\alpha \neq 1$ and set $\zeta = \zeta(\alpha, \beta) = -\beta \tan \frac{\pi}{2}$ and restricting to $u > 0$,

$$\begin{aligned} \frac{\phi(u) - 1}{(iu)^{p+1}} &= \left(\left[e^{-u \ (1+i\zeta)+i\delta u} - 1 \right] (-i) e^{-i(\pi/2)p} \right) u^{-p-1} \\ &= \left(-i \left(e^{-u} \ [\cos(\delta u - \zeta u \ ) + i \sin(\delta u - \zeta u \ )] - 1 \right) e^{-i(\pi/2)p} \right) u^{-p-1} \\ &= \left(\left[e^{-u} \ \sin(\delta u - \zeta u \ ) - i \left(e^{-u} \ \cos(\delta u - \zeta u \ ) - 1 \right) \right] \left[\cos\left(\tfrac{\pi p}{2}\right) - i \sin\left(\tfrac{\pi p}{2}\right) \right] \right) u^{-p-1} \end{aligned}$$

And therefore

$$\begin{aligned} \Re \frac{\phi(u) - 1}{(iu)^{p+1}} &= \left[\cos\left(\tfrac{\pi p}{2}\right) e^{-u} \ \sin(\delta u - \zeta u \ ) - \sin\left(\tfrac{\pi p}{2}\right) \left(e^{-u} \ \cos(\delta u - \zeta u \ ) - 1 \right) \right] u^{-p-1} \\ &= \cos\left(\tfrac{\pi p}{2}\right) \sin(\delta u - \zeta u \ ) u^{-p-1} e^{-u} \\ &\quad - \sin\left(\tfrac{\pi p}{2}\right) \left([\cos(\delta u - \zeta u \ ) - 1]\, u^{-p-1} e^{-u} \ + (e^{-u} \ - 1) u^{-p-1} \right) \end{aligned}$$

Integrating this from 0 to ∞, substituting $t = u$ in the last term to get

$$EX_+^p = \frac{\Gamma(p+1)}{\pi} \left[-\cos\left(\tfrac{\pi p}{2}\right) \widetilde{g}_{-p}(-\delta | \alpha, \beta) - \sin\left(\tfrac{\pi p}{2}\right) \{ g_{-p}(-\delta | \alpha, \beta) - \Gamma(1 - p/\alpha)/p \} \right].$$

Next consider $0 < p < \alpha = 1$. Use (3) again, so we need to simplify

$$\frac{\phi(u) - 1}{(iu)^{p+1}} = \left(\left[e^{-u(1+i \ \eta(u,1))+i\delta u} - 1 \right] (-i) e^{-i(\pi/2)p} \right) u^{-p-1}$$

$$
\begin{aligned}
&= \left(-i\left(e^{-u}\left[\cos(\delta u-\beta\eta(u,1))+i\sin(\delta u-\beta\eta(u,1))\right]-1\right)e^{-i(\pi/2)p}\right)u^{-p-1}\\
&= \left(\left[e^{-u}\sin(\delta u-\beta\eta(u,1))-i\left(e^{-u}\cos(\delta u-\beta\eta(u,1))-1\right)\right]\right.\\
&\qquad \left.\times\left[\cos\left(\tfrac{\pi p}{2}\right)-i\sin\left(\tfrac{\pi p}{2}\right)\right]\right)u^{-p-1}\\
\Re\frac{\phi(u)-1}{(iu)^{p+1}} &= \Big[\cos\left(\tfrac{\pi p}{2}\right)e^{-u}\sin(\delta u-\beta\eta(u,1))\\
&\qquad -\sin\left(\tfrac{\pi p}{2}\right)\left(e^{-u}(\cos(\delta u-\beta\eta(u,1))-1)+(e^{u}-1)\right)\Big]u^{-p-1}.
\end{aligned}
$$

Integrating from 0 to ∞ yields

$$
EX_+^p=\frac{\Gamma(p+1)}{\pi}\left[-\cos\left(\tfrac{\pi p}{2}\right)\widetilde{g}_{-p}(-\delta|1,\beta)-\sin\left(\tfrac{\pi p}{2}\right)\{g_{-p}(-\delta|1,\beta)-\Gamma(1-p)/p\}\right].
$$

When $p=1<\alpha<2$, EX exists and is equal to δ. Using Corollary 2 of Pinelis (2011) with $k=1$, $\ell=0$ shows

$$
EX_+=\frac{1}{2}EX+\frac{\Gamma(2)}{\pi}\int_0^\infty \Re\frac{\phi(u)-1}{(iu)^{p+1}}du=\frac{\delta}{2}+\frac{1}{\pi}\int_0^\infty \Re\frac{\phi(u)-1}{(iu)^{p+1}}du.
$$

The integrand is the same as above, with $\cos\left(\frac{\pi p}{2}\right)=0$ and $\sin\left(\frac{\pi p}{2}\right)=1$, so

$$
EX_+=\frac{\delta}{2}-\frac{1}{\pi}\left[g_{-1}(-\delta|\alpha,\beta)-\Gamma(1-1/\alpha)\right].
$$

When $1<p<\alpha<2$, Corollary 2 of Pinelis (2011) with $k=\ell=1$ shows

$$
EX_+^p=\frac{\Gamma(p+1)}{\pi}\int_0^\infty \Re\frac{\phi(u)-1-iuEX}{(iu)^{p+1}}du, \tag{4}
$$

Since $\alpha>1$, EX exists and is equal to δ. As above, for $u>0$,

$$
\begin{aligned}
\frac{\phi(u)-1-iu\delta}{(iu)^{p+1}} &= \left(\left[e^{-u\ \ (1+i\zeta u\ \ )+i\delta u}-1-i\delta u\right](-i)e^{-i(\pi/2)p}\right)u^{-p-1}\\
&= \left(-i\left(e^{-u}\ \ [\cos(\delta u-\zeta u\ \ )+i\sin(\delta u-\zeta u\ \ )]-1-i\delta u\right)e^{-i(\pi/2)p}\right)u^{-p-1}\\
&= \left[\left(e^{-u}\ \ \sin(\delta u-\zeta u\ \ )-\delta u\right)-i\left(e^{-u}\ \ \cos(\delta u-\zeta u\ \ )-1\right)\right]\\
&\qquad \times\left[\cos\left(\tfrac{\pi p}{2}\right)-i\sin\left(\tfrac{\pi p}{2}\right)\right]u^{-p-1}
\end{aligned}
$$

And therefore

$$
\begin{aligned}
\Re\frac{\phi(u)-1-iu\delta}{(iu)^{p+1}} &= \Big[\cos\left(\tfrac{\pi p}{2}\right)\left(e^{-u}\ \ \sin(\delta u-\zeta u\ \ )-\delta u\right)\\
&\qquad -\sin\left(\tfrac{\pi p}{2}\right)\left(e^{-u}\ \ \cos(\delta u-\zeta u\ \ )-1\right)\Big]u^{-p-1}\\
&= \cos\left(\tfrac{\pi p}{2}\right)\left([\sin(\delta u-\zeta u\ \ )-\delta u]u^{-p-1}e^{-u}\ \ +\delta(e^{-u}\ \ -1)u^{-p}\right)\\
&\qquad -\sin\left(\tfrac{\pi p}{2}\right)\left([\cos(\delta u-\zeta u\ \ )-1]\,u^{-p-1}e^{-u}\ \ +(e^{-u}\ \ -1)u^{-p-1}\right)
\end{aligned}
$$

Plugging this into (4) and integrating yields

$$EX_+^p = \frac{\Gamma(p+1)}{\pi}\Big\{\cos\left(\tfrac{\pi p}{2}\right)\left[-\widetilde{g}_{-p}(-\delta|\alpha,\beta)) + (\delta/\alpha)(\Gamma((1-p)/\alpha)\right] + \sin\left(\tfrac{\pi p}{2}\right)\left[-g_{-p}(-\delta|\alpha,\beta) + \Gamma(1-p/\alpha)/p\right]\Big\}$$

Now consider $\gamma \neq 1$. If $X \sim \mathbf{S}(\alpha,\beta,\gamma,\delta;1)$, then $X \overset{d}{=} \gamma Y$, where $Y \sim \mathbf{S}(\alpha,\beta,1,\delta^*;1)$, so $EX_+^p = \gamma^p EY_+^p$. In symbols,

$$m^p(\alpha,\beta,\gamma,\delta) = \gamma^p m^p(\alpha,\beta,1,\delta^*).$$

(b) This follows from $-X \sim \mathbf{S}(\alpha,-\beta,\gamma,-\delta;1)$. □

When $-1 < p < 0$, we conjecture that

$$m^p(\alpha,\beta,\gamma,\delta) = \gamma^p \frac{\Gamma(p+1)}{\pi}\left[-\sin\left(\tfrac{\pi p}{2}\right) g_{-p}(-\delta^*|\alpha,\beta) - \cos\left(\tfrac{\pi p}{2}\right)\widetilde{g}_{-p}(-\delta^*|\alpha,\beta)\right].$$

3. Related results

There are several corollaries to the preceding result. First, taking $p = 1$ in the previous result shows the following.

Corollary 2. *If $X \sim \mathbf{S}(\alpha,\beta,\gamma,\delta;1)$ with $\alpha > 1$, $-1 \le \beta \le 1$, $a \in \mathbb{R}$*

$$E(X-a)_+ = \frac{\delta - a}{2} + \frac{\gamma}{\pi}\left[\Gamma\left(1-\frac{1}{\alpha}\right) - g_{-1}\left(\frac{\delta-a}{\gamma}\ \alpha,\beta\right)\right].$$

Combining parts (a) and (b) of Theorem 1 yields.

Corollary 3. *If $X \sim \mathbf{S}(\alpha,\beta,\gamma,\delta;1)$ with $0 < \alpha < 2$, $-1 \le \beta \le 1$, $-1 \le p < \alpha$.*

$$E|X|^p = \gamma^p \frac{2\Gamma(p+1)}{\pi}\sin\left(\tfrac{\pi p}{2}\right)\left(\frac{\delta^*\Gamma(1-p/\alpha)}{p}\mathbb{1}_{\{p>0\}} - g_{-p}(-\delta^*|\alpha,\beta)\right)$$

$$EX^{<p>} = \gamma^p \frac{2\Gamma(p+1)}{\pi}\cos\left(\tfrac{\pi p}{2}\right)\left(\frac{\delta^*\Gamma((1-p)/\alpha)}{\alpha}\mathbb{1}_{\{p>1\}} - \widetilde{g}_{-p}(-\delta^*|\alpha,\beta)\right).$$

Proof $E|X|^p = EX_-^p + EX_+^p = m^p(\alpha,\beta,\gamma,\delta) + m^p(\alpha,-\beta,\gamma,-\delta)$ and $EX^{<p>} = m^p(\alpha,\beta,\gamma,\delta) - m^p(\alpha,-\beta,\gamma,-\delta)$. Use Theorem 1 and the reflection property: $g_d(-x|\alpha,\beta) = g_d(x|\alpha,-\beta)$. Note that as $p \to 0$, $E|X|^p \to E\,1 = 1$ and $EX^{<p>} \to -(2/\pi)\widetilde{g}_0(\delta^*|\alpha,\beta) = P(X>0) - P(X<0) = 1 - 2F(0)$. Also as $p \to 1$, $EX^{<p>} \to \delta$. □

In the strictly stable case, the expressions for EX_+^p can be simplified using closed form expressions for $g_d(0|\alpha,\beta)$ and $\widetilde{g}_d(0|\alpha,\beta)$ when $\alpha \neq 1$. To state the result, set

$$\theta_0 = \theta_0(\alpha,\beta) = \begin{cases} \alpha^{-1}\arctan\left(\beta\tan\frac{\pi}{2}\right) & \alpha \neq 1 \\ \pi/2 & \alpha = 1. \end{cases}$$

Lemma 4. *When $\alpha \neq 1$,*

$$g_d(0|\alpha,\beta) = \begin{cases} (\cos\alpha\theta_0)^{d/\alpha}\cos(d\,\theta_0)\Gamma(1+d/\alpha)/d & d>0 \\ (\ln(\cos\alpha\theta_0))/\alpha & d=0 \\ \left[(\cos\alpha\theta_0)^{d/\alpha}\cos(d\,\theta_0)-1\right]\Gamma(1+d/\alpha)/d & -\alpha<d<0 \end{cases}$$

$$\widetilde{g}_d(0|\alpha,\beta) = \begin{cases} -(\cos\alpha\theta_0)^{d/\alpha}\sin(d\,\theta_0)\Gamma(1+d/\alpha)/d & d\in(-\alpha,0)\cup(0,\infty) \\ -\theta_0 & d=0. \end{cases}$$

Proof Substitute $u = r$ in the expressions for $g_d(0|\alpha,\beta)$ and $\widetilde{g}_d(0|\alpha,\beta)$. Then use respectively the integrals 3.944.6, 3.948.2, 3.945.1, 3.944.5, and 3.948.1 pg. 492-493 of Gradshteyn and Ryzhik (2000). (Note that some of these formulas have mistyped exponents.) Finally, when $\alpha \neq 1$, $\alpha\theta_0 = -\arctan\zeta$, and for the allowable values of α and θ_0,

$$\cos\alpha\theta_0 = |\cos\alpha\theta_0| = (1+\tan^2\alpha\theta_0)^{-1/2} = (1+\zeta^2)^{-1/2}.$$

□

The following is a different proof of Theorem 2.6.3 of Zolotarev (1986).

Corollary 5. *Let X be strictly stable, e.g. $X \sim \mathbf{S}(\alpha,\beta,\gamma,0;1)$ with $\alpha \neq 1$ or ($\alpha = 1$ and $\beta = 0$) and $0 < p < \alpha$.*
(a) The fractional moment of the positive part of X is

$$EX_+^p = \frac{\gamma^p}{(\cos\alpha\theta_0)^{p/\alpha}}\frac{\Gamma(1-p/\alpha)}{\Gamma(1-p)}\frac{\sin p(\pi/2+\theta_0)}{\sin(p\pi)}.$$

(b) The fractional moment of the negative part of X is $EX_-^p = E(-X)_+^p$, which can be obtained from the right hand side above by replacing θ_0 with $-\theta_0$.

When $p = 1$, the product $\Gamma(1-p)\sin(\pi p)$ in the denominator above is interpreted as the limiting value as $p \to 1$, which is π.

Proof Note that when X is strictly stable, $\delta^* = 0$. First assume $0 < p < \min(1,\alpha)$ and substitute Lemma 4 into this case of Theorem 1

$$\begin{aligned} EX_+^p &= \frac{\gamma^p\Gamma(p+1)}{\pi}\left[\sin(\pi p/2)\left(\frac{\Gamma(1-p/\alpha)}{p} - \left((\cos\alpha\theta_0)^{-p/\alpha}\cos(-p\theta_0)-1\right)\frac{\Gamma(1-p/\alpha)}{-p}\right)\right. \\ &\qquad \left. -\cos(\pi p/2)(-\cos(\alpha\theta_0)^{-p/\alpha}\sin(-p\theta_0)\frac{\Gamma(1-p/\alpha)}{-p}\right] \\ &= \frac{\gamma^p\Gamma(p+1)\Gamma(1-p/\alpha)}{\pi p(\cos\alpha\theta_0)^{p/\alpha}}\left[\sin(\pi p/2)\cos(p\theta_0)+\cos(\pi p/2)\sin(p\theta_0)\right] \\ &= \frac{\gamma^p\Gamma(p+1)\Gamma(1-p/\alpha)}{\pi p(\cos\alpha\theta_0)^{p/\alpha}}\sin(\pi p/2+p\theta_0) \end{aligned}$$

Using the identity $\Gamma(p+1) = \pi p/(\Gamma(1-p)\sin p\pi)$ gives the result.

When $p = 1 < \alpha$, again using the appropriate part of Theorem 1 shows

$$\begin{aligned} EX_+ &= \gamma\left[0 + \frac{1}{\pi}\left(\Gamma(1-1/\alpha) - \left((\cos\alpha\theta_0)^{-1/\alpha}\cos(-\theta_0) - 1\right)\frac{\Gamma(1-1/\alpha)}{-1}\right)\right] \\ &= \frac{\gamma\Gamma(1-1/\alpha)}{\pi}\left[\cos\alpha\theta_0)^{-1/\alpha}\cos(\theta_0)\right]. \end{aligned}$$

When $1 < p < \alpha$, using Theorem 1 and $\delta^* = 0$,

$$\begin{aligned} EX_+^p &= \frac{\gamma^p\Gamma(p+1)}{\pi}\left[\sin(\pi p/2)\left(\frac{\Gamma(1-p/\alpha)}{p} - \left((\cos\alpha\theta_0)^{-p/\alpha}\cos(-p\theta_0) - 1\right)\frac{\Gamma(1-p/\alpha)}{-p}\right)\right. \\ &\qquad \left. + \cos(\pi p/2)\left(0 - \cos(\alpha\theta_0)^{-p/\alpha}\sin(-p\theta_0)\right)\frac{\Gamma(1-p/\alpha)}{-p}\right], \end{aligned}$$

and the rest is like the first case. □

The standard parameterization used above is discontinuous in the parameters near $\alpha = 1$, and it is not a scale-location family when $\alpha = 1$. To avoid this, a continuous parameterization that is a scale-location family can be used. We will say $X \sim \mathbf{S}(\alpha, \beta, \gamma, \delta; 0)$ if it has characteristic function

$$E\exp(iuX) = \begin{cases} \exp\left(-\gamma\ \ |u|\ \ \left[1 + i\beta(\tan\frac{\pi}{2})(\operatorname{sign} u)(|\gamma u|^{1-}\ \ - 1)\right] + i\delta u\right) & \alpha \neq 1 \\ \exp\left(-\gamma|u|\left[1 + i\beta(2/\pi)(\operatorname{sign} u)\log(\gamma|u|)\right] + i\delta u\right) & \alpha = 1. \end{cases}$$

A stable r. v. X can be expressed in both the 0-parameterization and the 1-parameterization, in which case the index α, the skewness β and the scale γ are the same. The only difference is in the location parameter: if X is simultaneously $\mathbf{S}(\alpha, \beta, \gamma, \delta_0; 0)$ and $\mathbf{S}(\alpha, \beta, \gamma, \delta_1; 1)$, then the shift parameters are related by

$$\delta_1 = \begin{cases} \delta_0 - \beta\gamma\tan\frac{\pi}{2} & \alpha \neq 1 \\ \delta_0 - (2/\pi)\beta\gamma\log\gamma & \alpha = 1. \end{cases}$$

Therefore, if $X \sim \mathbf{S}(\alpha, \beta, \gamma, \delta_0; 0)$,

$$EX_+^p = \begin{cases} m^p(\alpha, \beta, \gamma, \delta_0 - \beta\gamma\tan\frac{\pi}{2}) & \alpha \neq 1 \\ m^p(\alpha, \beta, \gamma, \delta_0 - (2/\pi)\beta\gamma\log\gamma) & \alpha = 1. \end{cases}$$

This quantity is continuous in all parameters.

For the above expressions for EX_+^p to be of practical use, one must be able to evaluate $g_d(\cdot|\alpha, \beta)$ and $\widetilde{g}_d(\cdot|\alpha, \beta)$. When d is a nonnegative integer, Nolan (2017) gives Zolotarev type integral expressions for these functions. However, this is not helpful here, where negative, non-integer values of d are needed. We have written a short R program to numerically evaluate the defining integrals for $g_d(\cdot|\alpha, \beta)$ and $\widetilde{g}_d(\cdot|\alpha, \beta)$. A single evaluation takes less than 0.0002 seconds on a modern desktop. This faster than numerically evaluating $EX_+^p = \int_0^\infty x^p f(x|\alpha, \beta, \gamma, \delta)dx$, because the latter requires many numerical calculations of the density $f(x|\alpha, \beta, \gamma, \delta)$.

References

References

Abdul-Hamid, H., Nolan, J. P., 1998. Multivariate stable densities as functions of one dimensional projections. J. Multivar. Anal. 67, 80–89.

Gradshteyn, I., Ryzhik, I., 2000. Table of Integrals, Series, and Products. Academic Press.

Nolan, J. P., 2017. Multivariate stable cumulative probabilities in polar form and related functions, in progress.

Pinelis, I., 2011. Positive-part moments via the Fourier-Laplace transform. Journal of Theoretical Probability 24 (2), 409–421.
URL `http://dx.doi.org/10.1007/s10959-010-0276-9`

Samorodnitsky, G., Taqqu, M., 1994. Stable Non-Gaussian Random Processes. Chapman and Hall, New York.

Zolotarev, V. M., 1986. One-dimensional Stable Distributions. Vol. 65 of Translations of Mathematical Monographs. American Mathematical Society, Translation of the original 1983 Russian edition.