On the Behavior of EMD and MEMD in Presence of Symmetric α-Stable Noise

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***Abstract*—Empirical Mode Decomposition (EMD) and its extended versions such as Multivariate EMD (MEMD) are data-driven techniques that represent nonlinear and non-stationary data as a sum of a finite zero-mean AM-FM components referred to as Intrinsic Mode Functions (IMFs). The aim of this work is to analyze the behavior of EMD and MEMD in stochastic situations involving non-Gaussian noise, more precisely, we examine the case of Symmetric α-Stable (SαS) noise. We report numerical experiments supporting the claim that both EMD and MEMD act, essentially, as filter banks on each channel of the input signal in the case of SαS noise. Reported results show that, unlike EMD, MEMD has the ability to align common frequency modes across multiple channels in same index IMFs. Further, simulations show that, contrary to EMD, for MEMD the stability property is well satisfied for the modes of lower indices and this result is exploited for the estimation of the stability index of the SαS input signal.**

***Index Terms*—EMD, filter banks, MEMD, symmetric α–stable noise.**

I. Introduction

EMPIRICAL MODE DECOMPOSITION (EMD) is a fully adaptive data-driven approach for the decomposition of non-stationary signals [1]. This technique decomposes any signal into a linear combination of a finite number of basis functions called intrinsic mode functions (IMFs). Being proven efficient when dealing with deterministic signals of oscillatory nature, EMD also reveals interesting properties when dealing with random signals. Dealing with such signals, their properties, their transformations, and their characterization in time and frequency domains has gained enormous attention in the last decade. Since a random signal is not repeatable in a predictable manner, it may only be described probabilistically or in terms of its average behavior. To be able to devise mathematical tools for this purpose, one needs to assume a statistical model which best describes the data. Evaluation of the performances of such methods depends upon the ability to determine the probability density function (pdf) of a function of the data samples, either analytically or numerically. When this is not possible, one must resort to Monte Carlo computer simulations. Among various probability distributions, the Gaussian distribution plays a predominant role in signal processing [2]. Many of the theorems of communications, estimation and detection theory have been formulated based on the Gaussian assumption thanks to the Central Limit Theorem (CLT), which holds for a large variety of distributions. Unfortunately, a broad class of phenomena encountered in practice are undeniably non-Gaussian and can be characterized by their impulsive nature [3]. Random fluctuations of gravitational fields, underwater acoustic noise of snapping shrimp, radar clutter, economic market indexes, Internet traffic or man-made noise have been found to belong to this class. Signals of this class are more likely to exhibit sharp spikes or bursts of outlying measurements than one would expect from normally distributed signals [4]–[6]. Impulsive perturbations of these signals are commonly modeled by symmetric α–stable (SαS) distributions. More precisely S α S distribution describes a large class of impulsive random variables with heavy-tailed distributions. This family possesses strong theoretical justifications according to the Generalized CLT (GCLT) which extends the CLT to the case when the summands are heavy-tailed [7]. Up to now the behavior of EMD has been analyzed in presence of fractional Gaussian noise (fGn) [8] and its extended version, Multivariate EMD (MEMD) [9], in white Gaussian noise case [10]. But much less attention has been paid to situations involving processes that generate impulsive signals or noise bursts using such decompositions. Thus, for more real world applications, it is important to investigate how such signal decompositions behave in the presence of SαS noise. Because Gaussian and stable non-Gaussian distributions are invariant under linear operations, they are very important in signal processing. Hence the importance of studying their characteristics when decomposed using EMD and MEMD.

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II. Basics of EMD and MEMD

EMD: Standard EMD breaks down any real-valued signal $x(t)$ into a reduced number of oscillating modes (AM-FM) called Intrinsic Mode Functions (IMFs) and a residual $r(t)$ consisting of all local trends [1]. By construction, each IMF is a zero-mean waveform whose number of zero-crossings (ZCs) differs at most by one from the number of its extrema. More precisely, EMD ends up with an expansion of the form:

$$x(t) = \sum_{m=1}^{M} c_m(t) + r(t) \tag{1}$$

where $c_m(t)$ is the mth IMF and M is the number of extracted modes. The number of extrema of $x(t)$ is decreased when going from one residual to the next.

MEMD: Standard EMD considers only 1D signals and the local mean is calculated by averaging the upper and lower envelopes obtained by interpolating between the local maxima and minima respectively. MEMD has been developed to process a general class of multivariate signals having an arbitrary number of channels [9]. For an n–dimensional signal the local mean cannot be defined directly, and thus the multiple n–dimensional envelopes are generated by projecting the signal along different directions in n–variate spaces. The calculation of the local mean can be considered as an approximation of the integral of all the envelopes along multiple directions in an n–dimension space. MEMD uses a vector-valued form of (1) to decompose a n–variate signal $\mathbf{x}(t)$ as follows:

$$\mathbf{x}(t) = \sum_{m=1}^{M} \mathbf{c}_m(t) + \mathbf{r}(t) \tag{2}$$

where M is the number of extracted n–variate modes, $\{\mathbf{c}_m\}_{m=1}^{M}$ contains scale-aligned intrinsic joint rotational modes and $\mathbf{r}(\mathrm{t}$ is the n–variate residue.

III. SαS Distributions

There is no closed-form for the pdf of the SαS distribution, but it is represented by its characteristic function: $\phi(\theta) = \exp(j\delta\theta - \gamma|\theta|^{\alpha})$, where δ is the location parameter and $\alpha \in (0, 2]$ is called the stability index. α is the most important parameter of the SαS distribution because it controls the density's tail heaviness and γ is the dispersion parameter that controls the width of the bell curve [4]. This scale parameter, similar to variance of the Gaussian distribution, determines the spread of the distribution around δ. The bell curve's tails get thicker as α falls from 2 to near 0. For $\alpha \in (1, 2]$, δ corresponds to the mean of the Sα S distribution while for $0 < \alpha \leq 1$, δ corresponds to its median. The only known closed-form Sα S pdfs are the thin-tailed Gaussian with $\alpha = 2$ (less impulsive) and the thick-tailed Cauchy with $\alpha = 1$ (more impulsive). The SαS distribution is the only distribution that verifies the GCLT:

GCLT: X is α-stable if and only if X is the limit in distribution of the sum $S_n = \frac{X_1+X_2+\ldots+X_n}{a_n} - b_n$, where $X_1, X_2, \ldots X_n$, are independent and identically distributed (i.i.d.) r.v.'s, $n \to \infty$, $b_n \in \mathbb{R}$ and $a_n \in \mathbb{N}$

Therefore the α-stable distribution is more general than the Gaussian distribution, and has a stronger justification since it covers the class of signals that does not satisfy the classical CLT.

IV. Properties in Presence of SαS Noise

A. Filter Bank Structure

A quantitative appreciation of the filter bank structure of EMD and MEMD, in presence of Sα S noise, can be done by measuring the mean frequency content of each IMF. Measuring the average number of ZCs is a meaningful way of characterizing its mean frequency, and the way this varies from mode to mode is an indication of the hierarchical structure of the filter bank [8]. Thus, it has been shown, by analyzing the graphical representation of the average number of ZCs as function of the IMF index, that output of EMD exhibits a dyadic filter-bank structure for fGn [8] and that MEMD acts as a dyadic filter-bank for multivariate white noise input [9]. Fig. 1 shows that this property also holds for a SαS i.i.d. process decomposed into IMFs by MEMD and EMD. For both EMD and MEMD, the standard stopping criterion described in [20] is used. The SαS noise is generated using the MatLab STABLE toolbox [21]. The number of channels of MEMD is set to 16 and the number of realizations is set to 1000. EMD is applied to each channel separately. For sake of readability, only the curves corresponding to 6 values of α are plotted. As evidenced in this figure, the logarithm (base 2) of the average number of ZCs is a decreasing function of the mode index m for both decompositions. For MEMD this figure suggests a functional relation of the form: $Z_\alpha[m] \propto \rho^{-m}$, with ρ close to 1.8. While for EMD the corresponding relation depends on the stability index α as follows:

$$Z_\alpha[m] \propto \rho_\alpha^{-m} \tag{3}$$

where $\rho_\alpha = \beta 2^{C\alpha}$, $\beta = 2^{1-2C}$ and C is a constant step equal to 0.2. Fig. 1 reveals similar results to those obtained in presence of fGn [8], where a slope of - 1 means an ideal dyadic filter-bank structure, indicating a quasi-dyadic filter bank structure for EMD and MEMD when facing a SαS i.i.d. process. Although EMD produces a more pronounced dyadic filter than MEMD for $\alpha \in]1.2; 2]$, they have almost the same slope for $\alpha \approx 1.2$ and MEMD becomes more dyadic for $\alpha \in [1; 1.2[$. It is also worth noticing that the slope of the line in the MEMD case is ≈ -0.84 and is the same regardless of the coefficient α, while in the EMD case, the slope varies from - 1 in the Gaussian case ($\alpha = 2$) to -0.80 ($\alpha = 1$).

B. Mode Alignment Property

The Hilbert transform produces meaningful instantaneous frequency (IF) only for mono-component data. Thus, accurate IF estimation requires that extracted modes by EMD or its extended versions be mono-component and locally orthogonal. A cue to identify such modes is to look at the cross-correlation (CC) coefficients between modes [16]. The schematic representation of CC values (correlograms) allows us to check if IMFs extracted from the input signal are aligned and have the same information at the same level of decomposition. It should be noted that mode alignment corresponds to finding a set of common scales/IMFs across different components (variates) of a multivariate signal, thus ensuring that the modes are matched both in the number and in scale properties [17]. The higher the CC, the less significant the splitting in separate IMFs. Thus, CC between normalized IMFs (leakage between sub-bands) may cause blurred time-frequency estimates such as IF. Using this quantitative evaluation, it has been shown that EMD and MEMD generate approximately mono-component and locally orthogonal data-driven basis functions in presence of white Gaussian noise ($\alpha = 2$) [12]. Fig. 2 shows CC of IMFs averaged over $J = 1000$ realizations of Sα S 2-channel process of length $L = 1000$ samples using EMD (channel-wise) and

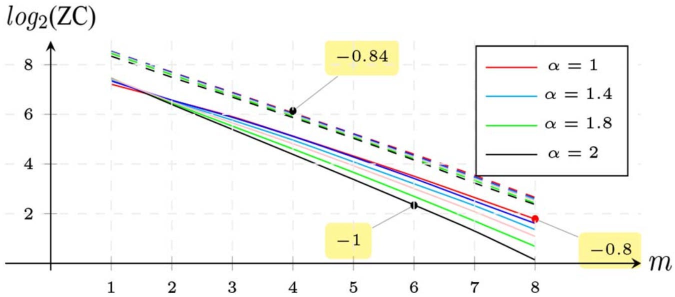

Fig. 1. Logarithm of average number of ZCs as a function of m. Slopes of the fitted lines are approximately equal to - 0.84 when using MEMD (dashed lines), and varying from - 1 ($\alpha = 2$) to - 0.8 ($\alpha = 1$) when using EMD (solid lines). For sake of readability MEMD curves are shifted upwards by one unit.

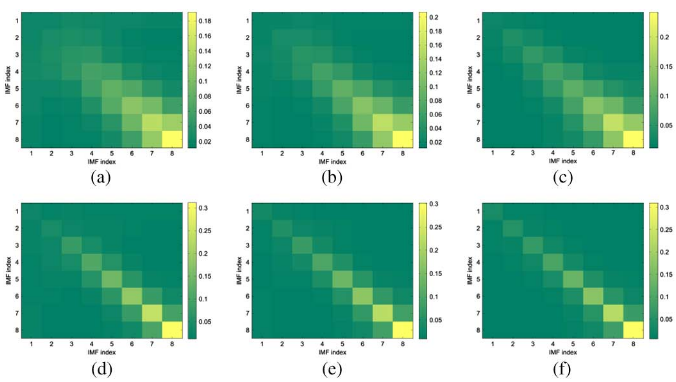

Fig. 2. CCs of IMFs for a bivariate SαS distribution using (a-c) EMD (channel-wise) and (d-f) MEMD.

MEMD. The CC estimates $\Theta[m, m']$ are calculated for IMFs obtained from MEMD and EMD as follows:

$$\Theta[m, m'] := \left| \frac{1}{J} \sum_{j=1}^{J} \frac{\Theta^j[m, m']}{\sqrt{\Theta^j[m, m]\Theta^j[m', m']}} \right|$$
$$\Theta^j[m, m'] := \frac{1}{L} \sum_{k=1}^{L} c_m^j(k) c_{m'}^j(k) \tag{4}$$

By definition, we have $0 \leq \Theta[m, m'] \leq 1$.

We report in Fig. 2 alignment results of three typical values of α. Fig. 2(d)–2(f) show that, on average, MEMD has almost the same behavior for all $\alpha \in [1, 2]$. Larger values along the diagonal ($m = m'$) suggest that the IMFs in MEMD are well aligned. For $\alpha = 1.8$ and $\alpha = 1.4$, both decompositions produce correlograms with diagonal-dominant elements while being more pronounced in the case of MEMD. For $\alpha = 1$, unlike MEMD, EMD does not exhibit a pronounced diagonal dominance, concluding that EMD does not produce same index IMFs with the same scale when α deviates from 2. As shown in Fig. 2(c), for more impulsive cases, significant values of CC estimates are observed off-diagonal ($m \neq m'$) indicating missaligned IMFs. This suggests that standard EMD is not well suited for decomposing signals of high impulsive nature.

C. Stability Test

EMD or MEMD are data-driven projections of a signal on some space, thus it is important to check if the stability property is preserved or not using these decompositions. As with any other family of distributions, it is not possible to prove that a given set is or is not stable, even for normality this is still an active research field [11]. A solution to this problem is to check whether or not data are consistent with stability hypothesis. More precisely, for plausibly stable smoothed density of data (Fig. 3(a),3(c)) the fitted distribution is compared to data using Quantile-Quantile (Q-Q) plot as shown in Figs. 3(b) and 3(d). Q-Q plot is designed to show the closeness of two distributions [11]. If the fitting is consistent, stability parameters $(\alpha, \beta, \gamma, \delta)$ are estimated. Four methods are used: Maximum Likelihood (ML) [19], Quantile, Empirical Characteristic Function (ECF) and Fractional Lower Order Moments (FLOM). Developing these estimation methods goes beyond the scope of this paper, and the reader is referred to [4], [13]–[15] and [19] for more details. If the estimates $(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta})$ differ significantly, the data are considered not stably distributed. While non-Gaussian stable distributions are heavy-tailed, most heavy-tailed distributions are not stable. In many cases, it is not appropriate to fit heavy-tailed data with a stable distribution. As shown in Fig. 3, the pdf of the first extracted IMF (averaged over 150 realizations) by EMD is bimodal, while the corresponding one of MEMD is unimodal. However, in both cases, an α-stable fitting is used to approximate the first mode even though, for the EMD case, this fitting is not accurate (one cannot fit a bimodal distribution using a unimodal stable one). Nevertheless, this test was made to prove that, even if the first IMF is not *stable*, when fitted using a *stable* distribution, its estimate $\hat{\alpha}$ is approximately equal to the original index α. The Q-Q plot of Fig. 3 shows that this fitting is more consistent in MEMD than in EMD. However, estimates $(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta})$, averaged over 150 realizations, reported in Table I for the first IMF results in a significantly different results using the four methods in the case of EMD, but the parameters retrieved in MEDM are on the average the same and close to the true parameters $(\alpha = 1.5, \beta = 0, \gamma = 1, \delta = 0)$. Therefore, these results support the claim that, in first approximation,

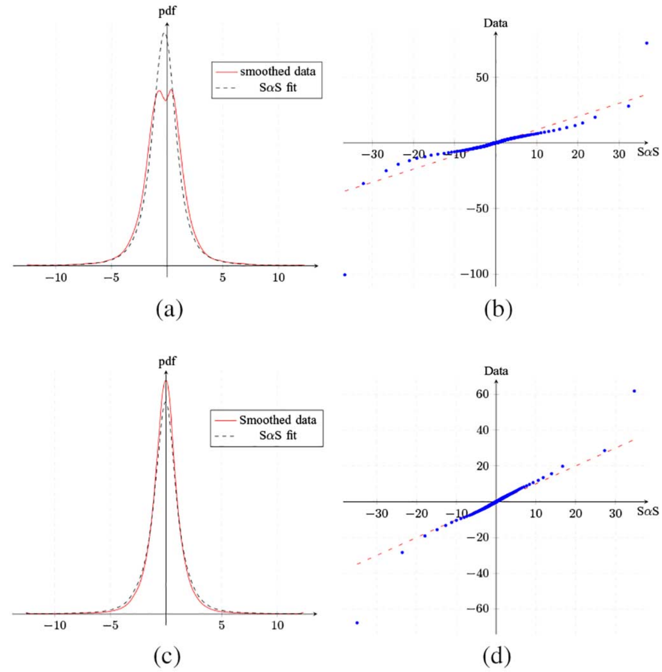

Fig. 3. Density plot and Q-Q plot of the first IMF for a SαS i.i.d. signal with $\alpha = 1.5$ and a data length of 10000 samples.

TABLE I
ESTIMATED PARAMETERS $(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta})$ OF THE FIRST IMF FOR EMD AND MEMD USING ML QUANTILE, ECF, AND FLMO

	EMD	MEMD
ML	(1.52, 0.000, 1.21, 0.013)	(1.48, 0.000, 0.74, 0.002)
Quantile	(1.63, 0.101, 1.28, 0.019)	(1.49, 0.005, 0.73, 0.007)
ECF	(1.48, 0.046, 1.16, 0.001)	(1.50, 0.014, 0.73, 0.005)
FLOM	(1.80, 0.000, 1.56, 0.000)	(1.50, 0.000, 0.73, 0.000)

stability is more preserved by MEMD than by EMD and more particularly the stability index α.

D. Estimation of the Stability Index

The most important parameter for the stable family is the stability index α that can be estimated from observations [18]. We decompose an input SαS noise with α values ranging from 1 to 2 into IMFs by EMD and MEMD. For sake of clarity, only five typical values of α are presented. Estimates $\hat{\alpha}$, using the method developed by McCulloch [14], are plotted as function of m in Fig. 4. As evidenced in Fig. 4, the stability property is mostly satisfied for modes of lower indices ($m \leq 5$) in MEMD, which capture the sharp spikes and tail heaviness of the original data. These modes, when fitted to a stable distribution, have the same α parameter as the input SαS signal. Thus satisfaction of this property, for such modes, allows us to estimate the α parameter of the input SαS signal. Note that for higher indices ($m \geq 5$), but with $\alpha < 1.6$, the stability is moderately satisfied. When going from the last IMFs to the residue, the distribution of these modes approaches a Gaussian distribution. For EMD, the stability property only holds for $\alpha = 2$ (white noise). We illustrate the relevance and the importance of our study on a real underwater acoustic signal containing: background underwater noise, propeller noise, Dolphin's sounds and sonar pings (Fig. 5). On can notice that outliers occurred more than frequently in this signal. Thus, adopting the Gaussian model is not relevant in such case. One way to study the statistical model of the signal is to decompose the signal into blocks, and estimate a statistical model for each block. It should be noted that this signal is 6 million samples long. We decompose it into blocks of length 10000 samples each (if we take fewer samples per block, the estimation of the pdf will not be accurate, and a larger block size will mitigate the effects of large spikes). Then α is estimated on each block and if the block could be modeled as Gaussian, then α should be close to 2, otherwise α will deviate from 2. The ML estimation is plotted in Fig. 5 (dashed red). However, when EMD is applied to each block before the estimation, then ML estimation is performed using only the first 4 modes, the result is plotted in solid blue. It can be seen that using EMD, the estimator captures almost all the regions where the signal experience non-Gaussian phenomenon. Thus, EMD can be very useful for such situations, where the data contains α–stable distribution along with other types of distributions (Gaussian or simply deterministic). In such cases, applying classical estimation techniques such as the ML on the whole data is not efficient because the α–stable distribution presence will be attenuated by the presence of other signals or distributions.

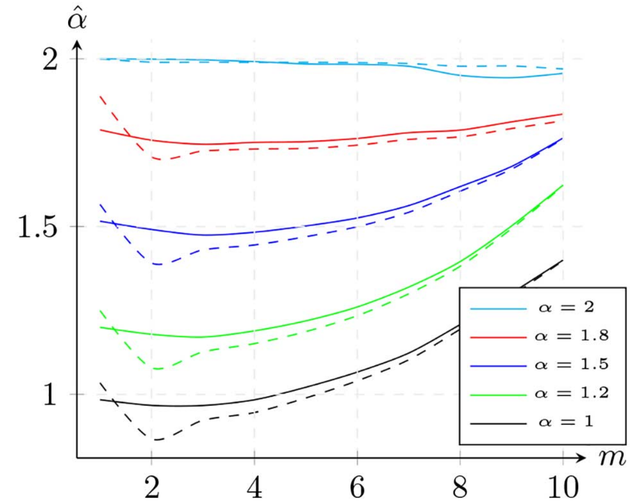

Fig. 4. $\hat{\alpha}$ values for input signal with different α (EMD: dashed lines- MEMD: solid lines).

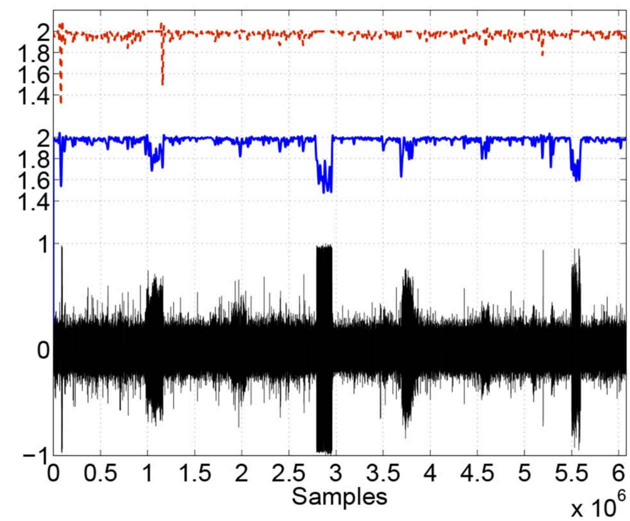

Fig. 5. Underwater acoustic signal (black) with α estimation per block using ML (dashed red) and EMD-ML (solid blue).

V. CONCLUSIONS

In this work we report on numerical experiments aimed at supporting the claim that in presence of SαS noise, both EMD and MEMD can be interpreted as filter bank on each channel of this process. Moreover, the first modes extracted by MEMD could be accurately fitted using an α–stable distribution, unlike original EMD, which produces bimodal modes that could not be fitted using a stable distribution. Unlike EMD, for MEMD the stability property is well satisfied for the modes of lower indices and this result is a new MEMD-based estimator of the stability index α of the SαS input signal. The reported results also show that MEMD aligns similar modes present across multiple channels in same-index IMFs for varying values of the stability index α. This property is crucial for real world applications such as Instantaneous Frequency estimation, signals denoising or data fusion. However, mode alignment is not achieved by standard EMD applied channel-wise and thus is not well suited for decomposing signals of high impulsive nature (small α values). As future work we plan to study the behavior of MEMD and EMD with isotropic, elliptical and other multivariate stable processes.

REFERENCES

[1] N. E. Huang, Z. Shen, S. Long, M. Wu, H. Shih, Q. Zheng, N. Yen, C. Tung, and H. Liu, "The empirical mode decomposition and hilbert spectrum for non-linear and non-stationary time series analysis," in *Proc. Roy. Soc. A*, 1998, vol. 454, pp. 903–995.

[2] M. Vetterli, J. Kovacevic, and V. K. Goyal, *Foundations of Signal Processing*. Cambridge, U.K.: Cambridge Univ. Press, 2014.

[3] M. Shao and C. L. Nikias, "Signal processing with fractional lower order moments: Stable processes and their applications," *Proc. IEEE*, vol. 81, no. 7, pp. 986–1010, 1993.

[4] C. L. Nikias and M. Shao, *Signal Processing with α–Stable Distributions and Applications*. New York, NY, USA: Wiley, 1995.

[5] M. Sahmoudi, K. Abed-Meraim, and M. Benidir, "Blind separation of impulsive alpha-stable sources using minimum dispersion criterion," *IEEE Signal Process. Lett.*, vol. 12, no. 4, pp. 281–284, 2005.

[6] F. X. Socheleau and D. Pastor, "On symmetric alpha-stable noise after short-time fourier transformation," *IEEE Signal Process. Lett.*, vol. 20, no. 5, pp. 455–458, 2013.

[7] G. Samorodnitsky and M. S. Taqqu, *Stable non-Gaussian random processes: Stochastic models with infinite variance*. New York, NY, USA: Chapman & Hall, 1994.

[8] P. Flandrin, G. Rilling, and P. Goncalves, "Empirical mode decomposition as a filter bank," *IEEE Signal Process. Lett.*, vol. 11, no. 2, pp. 112–114, 2004.

[9] N. Rehman and D. P. Mandic, "Multivariate empirical mode decomposition," in *Proc. Roy. Soc. A*, 2010, vol. 466, no. 2117, pp. 1291–1302.

[10] N. Rehman and D. P. Mandic, "Filter bank property of multivariate empirical mode decomposition," *IEEE Trans. Signal Process.*, vol. 59, no. 5, pp. 2441–2426, 2011.

[11] J. P. Nolan, "Fitting data and assessing goodness of fit with stable distributions," in *Proc. Heavy Tails, Appliat. Heavy-Tailed Distributions Economics, Eng., Statist*, Washington, DC, USA, Jun. 1999, p. 42, P. TAILS.

[12] D. P. Mandic, N. Rehman, Z. Wu, and N. E. Huang, "Empirical mode decomposition-based time-frequency analysis of multivariate signals," *IEEE Signal Process. Mag.*, vol. 36, no. 6, p. 741786, 2013.

[13] J. H. McCulloch, "Maximum likelihood estimation of symmetric stable parameters," Tech. Rep., Dept. Economics, Ohio State Univ., Columbus, OH, USA, 1998, p. 145.

[14] J. H. McCulloch, "Simple consistent estimators of stable distribution parameters," *Commun. Statist. Simul. Comput.*, vol. 15, no. 5, pp. 1109–1136, 1986.

[15] S. M. Kogon and D. B. Williams, , R. Adler, R. Feldman, and M. Taqqu, Eds., *Characteristic function based estimation of stable parameters. in A Practical Guide to Heavy Tailed Data*. Boston, MA, USA: Birkhauser, 1998, pp. 311–338.

[16] P. Flandrin, P. Goncalves, and G. Rilling, , N. E. Huang and S. S. P. Shen, Eds., *EMD Equivalent Filter Banks, from Interpretation to Applications in Hilbert-Huang Transform and Its Applications* . Singapore: World Scientific, 2005, pp. 57–74.

[17] N. Rehman, C. Park, N. E. Huang, and D. P. Mandic, "EMD via MEMD: Multivariate noise-aided computation of standard EMD," *Adv. Adapt. Data Anal.*, vol. 5, no. 2, pp. 1–25, 2013.

[18] R. F. Brcich, D. R. Iskander, and A. M. Zoubir, "The stability test for symmetric alpha-stable distributions," *IEEE Trans. Signal Process.*, vol. 53, no. 3, pp. 977–986, 2005.

[19] J. P. Nolan, , O. E. Barndordd-Nielsen, T. Mikosch, and S. I. Resnick, Eds., *Maximum-Likelihood Estimation of Stable Parameters In Lévy Processes: Theory and Applications*. Boston, MA, USA: Birkhäuser, 2001, pp. 397–400.

[20] G. Rilling, P. Flandrin, and P. Gonçalvès, "On empirical mode decomposition and its algorithms," in *IEEE-EURASIP, Workshop on Nonlinear Signal and Image Processing, NSIP*, Jun. 2003.

[21] [Online]. Available: http://www.robustanalysis.com/