Extremes

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--Manuscript Draft--

Manuscript Number:	
Full Title:	A measure of dependence for stable distributions
Article Type:	Research Article
Keywords:	multivariate stable laws; independence
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Corresponding Author's Institution:	American University
Corresponding Author's Secondary Institution:	
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EXTREMES manuscript No.
(will be inserted by the editor)

A measure of independence for stable distributions

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Received: date / Accepted: date

Abstract A measure of dependence is proposed for stable distributions that completely characterizes independence for a bivariate stable distribution. Properties of this measure are analyzed, and contrasted with the covariation and co-difference. A sample analog of the measure is defined demonstrated on simulated and real data, including time series and distributions in the domain of attraction of a stable law.

Keywords Stable vectors · Dependence measure · Test statistic

Mathematics Subject Classification (2010) MSC 60E07 · MSC 60G52 · MSC 62H20

1 Introduction

A d-dimensional random vector $\mathbf{X} = (X_1, \ldots, X_d)$ is said to be stable if for all $n = 2, 3, 4, \ldots$, there is a constant $a_n > 0$ and a vector $\mathbf{b}_n \in \mathbb{R}^d$ such that $\mathbf{X}_1 + \mathbf{X}_2 + \cdots \mathbf{X}_n \stackrel{d}{=} a_n \mathbf{X} + \mathbf{b}_n$, where $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \ldots$ are i.i.d. copies of $\mathbf{X}$.

The second author was supported by an agreement with Cornell University, Operations Research & Information Engineering under W911NF-12-1-0385 from the Army Research Development and Engineering Command.

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A multivariate stable distribution is usually described by a spectral measure Λ, a finite Borel measure on the unit sphere $\mathbb{S} = \{\mathbf{s} \in \mathbb{R}^d : |\mathbf{s}| = 1\}$, and a shift vector $\boldsymbol{\delta} \in \mathbb{R}^d$. There are multiple parameterizations of stable laws; we will use two, which we call the 0-parameterization and the 1-parameterization. We will say $\mathbf{X} \sim \mathbf{S}(\alpha, \Lambda, \boldsymbol{\delta}; j)$, $j = 0, 1$ if its joint characteristic function is given by

$$\phi(\mathbf{u}) = E \exp(i\langle \mathbf{u}, \mathbf{X}\rangle) = \exp\left(- \int_{\mathbb{S}} \omega\left(\langle \mathbf{u}, \mathbf{s}\rangle | \alpha; j\right) \, \Lambda(d\mathbf{s}) + i\langle \mathbf{u}, \boldsymbol{\delta}\rangle \right),$$

where

$$\omega(t|\alpha; j) = \begin{cases} |t|^\alpha [1 + i \operatorname{sign}(t) \tan\frac{\pi\alpha}{2} (|t|^{1-\alpha} - 1)] & \alpha \neq 1, j = 0 \\ |t|^\alpha [1 - i \operatorname{sign}(t) \tan\frac{\pi\alpha}{2}] & \alpha \neq 1, j = 1 \\ |t| [1 + i \operatorname{sign}(t) \frac{2}{\pi} \log|t|] & \alpha = 1, j = 0, 1. \end{cases}$$

The 1-parameterization is more commonly used, but because $|\tan(\pi\alpha/2)| \to \infty$ as $\alpha \to 1$, the 1-parameterization is discontinuous in α. Since $\tan(\pi\alpha/2)(|t|^{1-\alpha} - 1) \to \frac{2}{\pi} \log|t|$ as $\alpha \to 1$, the 0-parameterization is a continuous parameterization of multivariate stable laws. If $\mathbf{X} \sim \mathbf{S}(\alpha, \Lambda, \boldsymbol{\delta}_0; 0)$ and $\mathbf{X} \sim \mathbf{S}(\alpha, \Lambda, \boldsymbol{\delta}_1; 1)$, then the shift vectors are related by

$$\boldsymbol{\delta}_1 = \begin{cases} \boldsymbol{\delta}_0 - \tan\frac{\pi\alpha}{2} \int_{\mathbb{S}} \mathbf{s} \, \Lambda(d\mathbf{s}) & \alpha \neq 1 \\ \boldsymbol{\delta}_0 & \alpha = 1. \end{cases}$$

Another way of describing a multivariate stable law is by the use of linear projections. If $\mathbf{X}$ is a stable vector, then every one dimensional projection $\langle \mathbf{u}, \mathbf{X}\rangle = u_1 X_1 + u_2 X_2 + \cdots + u_d X_d$ has a univariate stable distribution, with a constant index of stability α and skewness $\beta(\mathbf{u})$, scale $\gamma(\mathbf{u})$ and shift $\delta(\mathbf{u})$ that depend on the direction $\mathbf{u}$, see Samorodnitsky and Taqqu (1994), Section 2.1. (The converse is true if $\alpha \geq 1$; when $\alpha < 1$ an extra condition is needed for the converse, see the discussion after Lemma 4.1 of Nolan (2010).) We will call the functions $\beta(\cdot)$, $\gamma(\cdot)$ and $\delta(\cdot)$ the projection parameter functions. Since they uniquely determine all one dimensional projections, they determine the joint distribution via the Cramér-Wold device. In this case, we will parameterize $\mathbf{X}$ by these projection parameter functions: $\mathbf{X} \sim \mathbf{S}(\alpha, \beta(\cdot), \gamma(\cdot), \delta(\cdot); j)$, $j = 0$ or $j = 1$. It is well known that $\gamma(\mathbf{u}) = \left(\int_{\mathbb{S}} |\langle \mathbf{u}, \mathbf{s}\rangle|^\alpha \Lambda(d\mathbf{s})\right)^{1/\alpha}$ in both parameterizations. One fact we will note here is that these projection parameter functions have scaling properties, in particular $\gamma(r\mathbf{u}) = r\gamma(\mathbf{u})$, so knowing them on the unit sphere determines them everywhere. For the symmetric case, the joint characteristic function is $E \exp(i\langle \mathbf{u}, \mathbf{X}\rangle) = \exp(-\gamma^\alpha(\mathbf{u})) = \exp(-|\mathbf{u}|^\alpha \gamma^\alpha(\mathbf{u}/|\mathbf{u}|))$, so the values of $\gamma(\cdot)$ on $\mathbb{S}$ completely determines the joint distribution.

In Nolan (2010), the projection parameter functions were used to measure the distance between two multivariate stable distributions. Here we will adapt that idea to measure distance between a multivariate stable distribution and the stable law with independent components. Specifically, let $\mathbf{X} = (X_1, X_2)$ be a bivariate α-stable r. vector. We assume that the components are normalized: $\gamma_j = \gamma(\mathbf{e}_j) = 1$, $j = 1, 2$, where $\mathbf{e}_j$ is the j-th standard unit vector. Set $\gamma_\perp(\mathbf{u}) = (|u_1|^\alpha + |u_2|^\alpha)^{1/\alpha}$; this

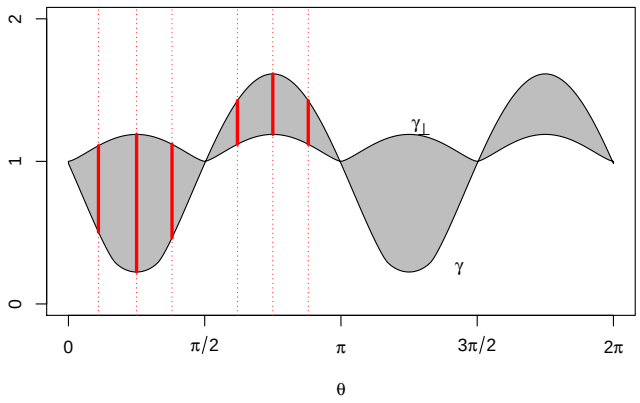

Fig. 1 η_1 is the area between the curves $\gamma(\cdot)$ and $\gamma_\perp(\cdot)$. The vertical lines are discussed when the sample estimator $\widehat{\eta}_p$ is defined below.

is the scale function of any two-dimensional stable distribution having independent components and unit scales. (The distribution can be symmetric or skewed, so there are multiple stable distributions that have this scale function. Specifying the skewness β_1 and β_2 for each component uniquely determines the joint distribution.) For any $p \in [1, \infty]$, define a measure of independence by

$$\eta_p = \eta_p(X_1, X_2) = \|\gamma^\alpha(u_1, u_2) - \gamma_\perp^\alpha(u_1, u_2)\|_{L^p(\mathbb{S}, d\mathbf{u})}. \tag{1}$$

Here $d\mathbf{u}$ is surface area on $\mathbb{S}$ (unnormalized, with total mass 2π). Figure 1 illustrates the geometric idea behind η_p.

The following simple result is the motivation for this definition.

Proposition 1 *Let $\mathbf{X} = (X_1, X_2)$ be an $\alpha-$stable random vector with normalized components, $\alpha \in (0, 2)$. Then $\mathbf{X}$ has independent components if and only if $\eta_p = 0$ for some (every) $p \in [1, \infty]$.*

Proof It is well-known that $\mathbf{X}$ has independent components if and only if the spectral measure is concentrated on the 4 points where the axes intersect the unit circle, e.g. Samorodnitsky and Taqqu (1994), Example 2.3.5. Let $\lambda_{j,+}$ be the spectral mass at $\mathbf{e}_j$ and $\lambda_{j,-}$ be the spectral mass at $-\mathbf{e}_j$, $j = 1, 2$. Then

$$\gamma^\alpha(\mathbf{u}) = |u_1|^\alpha(\lambda_{1,+} + \lambda_{1,-}) + |u_2|^\alpha(\lambda_{2,+} + \lambda_{2,-}).$$

Since $\gamma(\mathbf{e}_j) = \gamma_j = 1$, we must have $\gamma_j^\alpha = (\lambda_{j,+} + \lambda_{j,-}) = 1$, $j = 1, 2$. Thus $\gamma^\alpha(u_1, u_2) = |u_1|^\alpha + |u_2|^\alpha$, and therefore $\eta_p = 0$ for all p. For the converse, the scale function $\gamma(\cdot)$ is continuous, so the only way η_p can be 0 is if $\gamma^\alpha(u_1, u_2) = \gamma_\perp(u_1, u_2)$. The following argument shows that the spectral measure is concentrated on $\{\mathbf{e}_1, -\mathbf{e}_1, \mathbf{e}_2, -\mathbf{e}_2\}$. Define $\Lambda_{\text{sym}}(A) = (\Lambda(A) + \Lambda(-A))/2$. This is a symmetric

spectral measure with same scale function as Λ. By the uniqueness of the spectral measure, this means the spectral measure Λ_{sym} must be of the form $(\epsilon_{\mathbf{e}_1} + \epsilon_{-\mathbf{e}_1} + \epsilon_{\mathbf{e}_2} + \epsilon_{-\mathbf{e}_2})/2$, the spectral measure of the symmetric stable r. vector with the above scale function. By the definition of Λ_{sym}, the original Λ must also be concentrated on $\pm\mathbf{e}_j$, $j = 1, 2$. Thus $\mathbf{X}$ has independent components. □

Here are several comments about η_p and its properties.

1. The p-norm in (1) is evaluated as an integral over the unit circle $\mathbb{S}$, not all of $\mathbb{R}^2$. In polar coordinates,

$$\eta_p = \left(2\int_0^\pi |\gamma^\alpha(\cos\theta, \sin\theta) - \gamma_\perp^\alpha(\cos\theta, \sin\theta)|^p \, d\theta\right)^{1/p}, \tag{2}$$

   where the interval of integration has been reduced by using the fact that $\gamma(\cdot)$ is π-periodic. Below we will use polar notation for the scale function on the unit circle when it is more convenient: $\gamma(\theta) := \gamma(\cos\theta, \sin\theta)$. In particular, $\gamma_\perp(\theta) = (|\cos\theta|^\alpha + |\sin\theta|^\alpha)^{1/\alpha}$.
2. α can be any value in $(0, 2)$ and $\mathbf{X}$ can have symmetric or non-symmetric components, and it can be centered or shifted.
3. η_p is symmetric: $\eta_p(X_1, X_2) = \eta_p(X_2, X_1)$.
4. $\eta_p \geq 0$ by definition, so with this measure there is no notion of positive or negative dependence. Some authors have defined a signed measure of dependence, e.g. the signed covariation of Garel and Kodia (2014), but there is an arbitrariness with the sign. While this assignment may make sense in a some cases, e.g. elliptically contoured stable, it doesn't seem to make sense in general. For example, a distribution that is a rotation by $\pi/4$ of the bivariate independent stable component case has probability concentrated along both lines $y = x$ and $y = -x$; we do not see a meaningful way of assigning a sign to such dependence. See the discussion in Section 2 on the distribution of mass among different quadrants.
5. The definition makes sense in the Gaussian case: when $\alpha = 2$, the scale function for a bivariate Gaussian distribution with standardized marginals and correlation ρ is $\gamma^2(\mathbf{u}) = 1 + 2\rho u_1 u_2$ and $\gamma_\perp^2(\mathbf{u}) = 1$ on $\mathbb{S}$. Then $\eta_p^p = |2\rho|^p \int_{\mathbb{S}} |u_1 u_2|^p d\mathbf{u}$, so $\eta_p = k_p|\rho|$.
6. In the elliptically contoured/sub-Gaussian stable case with $0 < \alpha < 2$, and shape matrix $(1, \rho; \rho, 1)$, $\gamma^\alpha(\mathbf{u}) = (u_1^2 + 2\rho u_1 u_2 + u_2^2)^{\alpha/2}$, so

$$\eta_p^p = 2\int_0^\pi \left|(1 + 2\cos\theta\sin\theta)^{\alpha/2} - (|\cos\theta|^\alpha + |\sin\theta|^\alpha\right|^p d\theta.$$

   This can be computed numerically, see Figure 2 for a plot of η_2 in the elliptical case.
7. If the components of $\mathbf{X}$ are not standardized, then define

$$\eta_p(X_1, X_2) = \left\| \gamma^\alpha\left(\frac{u_1}{\gamma_1}, \frac{u_2}{\gamma_2}\right) - \gamma_\perp^\alpha(u_1, u_2) \right\|_{L^p(\mathbb{S}, d\mathbf{u})}.$$

$\gamma(u_1/\gamma_1, u_2/\gamma_2)$ is the scale function of the scaled r. vector $\mathbf{Y} = (X_1/\gamma_1, X_2/\gamma_2)$. This normalizes the measure of independence.

8. A stable random vector $\mathbf{X} = (X_1, \ldots, X_d)$ has mutually independent components if and only if all pairs are independent, e.g. Corollary 3.5.4 in Samorodnitsky and Taqqu (1994). Hence the components of $\mathbf{X}$ are mutually independent if and only if $\eta_p(X_i, X_j) = 0$ for all $i > j$.
9. Since $L^p(\mathbb{S}, d\mathbf{s})$ is a finite measure space, Holder's Inequality shows for $1 \leq p \leq \infty$, $\eta_1 \leq (2\pi)^{1+1/p}\eta_p$. Also, Section 2 below shows $|\gamma^\alpha(\mathbf{u}) - \gamma^\alpha_\perp(\mathbf{u})| \leq 2$, so for any $q > 1$, $\eta_q = \left(\int |\gamma^\alpha(\mathbf{u}) - \gamma^\alpha_\perp(\mathbf{u})|^q\right)^{1/q} \leq \left(2^{q-1}\int |\gamma^\alpha(\mathbf{u}) - \gamma^\alpha_\perp(\mathbf{u})|\right)^{1/q} = 2^{1-1/q}\eta_1^{1/q}$. Hence, η_p is small for some p if and only if it is small for all q. We will focus on the case $p = 2$ below, particularly in the sample analog $\widehat{\eta}_2$.
10. η_p measures how far the scale function of $\mathbf{X}$ is from the scale function of a stable r. vector with independent components. When $\mathbf{X}$ is symmetric, not only is is $\eta_p = 0$ a characterization of independence, but more generally the size of η_p is a measure of closeness between $\mathbf{X}$ and the independent case. Let $f(\cdot)$ be the density of $\mathbf{X}$ and let $f_\perp(\cdot)$ be the density of the independent components case (both having normalized components), then Nolan (2010) shows

$$\sup_{\mathbf{x}\in\mathbb{R}^2} |f(\mathbf{x}) - f_\perp(\mathbf{x})| \leq k_\alpha \|\gamma(\mathbf{u}) - \gamma_\perp(\mathbf{u})\|_1.$$

When $\alpha \in (0, 1]$, the right hand side can be bounded by η_p. To see this, Section 2 shows $\gamma(\mathbf{u})$ and $\gamma_\perp(\mathbf{u})$ are in a bounded interval $[0, R_\alpha]$. For α in the range (0,1] and $s, t \in [0, R_\alpha]$, $|s - t| \leq (R_\alpha^{1-\alpha}/\alpha)|s^\alpha - t^\alpha|$, so

$$\begin{aligned}\|\gamma(\mathbf{u}) - \gamma_\perp(\mathbf{u})\|_1 &= \int |\gamma(\mathbf{u}) - \gamma_\perp(\mathbf{u})| d\mathbf{s} \\ &\leq c_\alpha \int |\gamma^\alpha(\mathbf{u}) - \gamma^\alpha_\perp(\mathbf{u})| d\mathbf{s} = c_\alpha \eta_1 \leq c'_\alpha \eta_p.\end{aligned}$$

When $1 < \alpha \leq 2$, a similar result can be shown by modifying the arguments in Nolan (2010). In the non-symmetric case, showing that the respective densities $f(\mathbf{x})$ and $f_\perp(\mathbf{x})$ are close requires an additional condition, i.e. that $\|\beta(\mathbf{u}) - \beta_\perp(\mathbf{u})\|_1$ is also small.
11. The same idea can be used to compare the distribution given by $\gamma^\alpha(\cdot)$ to a different model, other than the independent one, by replacing $\gamma^\alpha_\perp(\cdot)$ with the appropriate scale function. For example, using $\gamma^\alpha_{iso}(\cdot) = 1$ in place of $\gamma^\alpha_\perp(\cdot)$ in (1) would measure distance from the isotropic distribution. Modification of the sample measure $\widehat{\eta}_2$ in Section 3 is straightforward.

The next section examines the scale function, the proposed measure η, and compares with covariation and co-difference. In the third section, a sample analogue is defined and demonstrated with simulated and real data with bivariate and multivariate data, time series and a modification is given for vectors in the domain of attraction of a stable law.

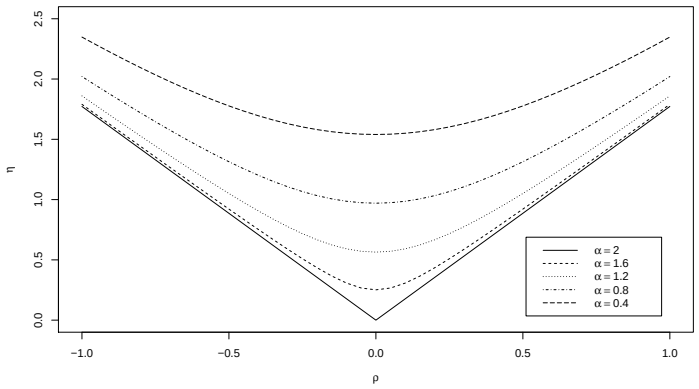

Fig. 2 Plot of η_2 for α-stable elliptically contoured stable distribution with shape parameter ρ.

2 Properties of $\gamma(\cdot)$, η_p, covariation and co-difference

In multivariate extreme value theory, Pickands (1981) defined a function that characterizes the dependence in a bivariate extreme value distribution. Here we briefly describe how the stable scale function can serve a similar purpose.

Let $\mathbf{X}$ be a bivariate α-stable r. vector with spectral measure Λ and scale function $\gamma(\cdot)$. Throughout this section we assume that $\mathbf{X}$ has normalized components: $\gamma_1 = \gamma_2 = 1$. As above, let $\gamma(\theta) = \gamma(\cos\theta, \sin\theta)$, $\theta \in [0, 2\pi]$. In polar coordinates,

$$\gamma^\alpha(\theta) = \int_0^{2\pi} |\langle(\cos\theta, \sin\theta), (\cos\phi, \sin\phi)\rangle|^\alpha \Lambda(\phi) = \int_0^{2\pi} |\cos(\theta - \phi)|^\alpha \Lambda(\phi).$$

First, let us look for envelope functions:

$$\gamma_{\min}(\theta) := \inf_\gamma \gamma(\theta) \quad \text{and} \quad \gamma_{\max}(\theta) := \sup_\gamma \gamma(\theta),$$

where the $\inf$ and $\sup$ are taken over all valid scale functions for a bivariate $\alpha-$stable random vector with normalized components. Proposition 2 gives an explicit formula for $\gamma_{\max}(\theta)$, Proposition 3 gives a candidate for $\gamma_{\min}(\theta)$; both depend only on α.

Proposition 2 *For normalized components,* $0 \le \gamma(\theta) \le \gamma_{\max}(\theta)$*, where*

$$\gamma_{\max}(\theta) = \begin{cases} |\cos\theta| + |\sin\theta| & 1 \le \alpha < 2 \\ (|\cos\theta|^\alpha + |\sin\theta|^\alpha)^{1/\alpha} & 0 < \alpha < 1. \end{cases}$$

Proof Since the distribution has normalized components, $\gamma(1,0) = \gamma(0,1) = 1$. The proof is easier using rectangular coordinates: write $\mathbf{u} \in \mathbb{S}$ as $(u_1, u_2) = (\cos\theta, \sin\theta)$.

When $0 < \alpha \leq 1$,

$$\begin{aligned}\gamma^\alpha(u_1, u_2) &= \int_{\mathbb{S}} |u_1 s_1 + u_2 s_2|^\alpha \Lambda(d\mathbf{s}) \\ &\leq \int_{\mathbb{S}} |u_1 s_1|^\alpha \Lambda(d\mathbf{s}) + \int_{\mathbb{S}} |u_2 s_2|^\alpha \Lambda(d\mathbf{s}) \\ &= |u_1|^\alpha \gamma^\alpha(1,0) + |u_2|^\alpha \gamma^\alpha(0,1) = |u_1|^\alpha + |u_2|^\alpha = \gamma_\perp^\alpha(u_1, u_2)\end{aligned}$$

When $\alpha \geq 1$, using the triangle inequality,

$$\begin{aligned}\gamma(u_1, u_2) &= \left(\int_{\mathbb{S}} |u_1 s_1 + u_2 s_2|^\alpha \Lambda(d\mathbf{s}) \right)^{1/\alpha} \\ &\leq \left(\int_{\mathbb{S}} |u_1 s_1|^\alpha \Lambda(d\mathbf{s}) \right)^{1/\alpha} + \left(\int_{\mathbb{S}} |u_2 s_2|^\alpha \Lambda(d\mathbf{s}) \right)^{1/\alpha} \\ &= |u_1| \gamma(1,0) + |u_2| \gamma(0,1) = |u_1| + |u_2|\end{aligned}$$

The expressions for $\gamma_{\max}(\cdot)$ are sharp. To see this, set

$$\begin{aligned}\gamma_+(\theta) &= 2^{1/2} |\cos(\theta - \pi/4)| \\ \gamma_-(\theta) &= 2^{1/2} |\cos(\theta - 3\pi/4)| \\ \gamma_\perp(\theta) &= (|\cos\theta|^\alpha + |\sin\theta|^\alpha)^{1/\alpha}\end{aligned}$$

Note that $\gamma_+(\cdot)$ corresponds to a bivariate stable distribution with exact positive linear dependence ($X_2 = X_1$, spectral measure with mass of weight $2^{\alpha/2}$ on the diagonal), $\gamma_-(\cdot)$ corresponds to exact negative linear dependence ($X_2 = -X_1$, mass of weight $2^{\alpha/2}$ on the anti-diagonal), and $\gamma_\perp(\cdot)$ corresponds independent components (masses of weight 1 on the line $y = 0$ and weight 1 on the line $x = 0$). When $\alpha \geq 1$, $|\cos\theta| + |\sin\theta| = \max(\gamma_-(\theta), \gamma_+(\theta))$, so the upper bound is achieved by $\gamma_-(\theta)$ in the first and third quadrants and $\gamma_+(\theta)$ in the second and fourth quadrants. When $\alpha \leq 1$, the upper bound is achieved by the independent component case. □

We conjecture that a sharp lower bound $\gamma_{\min}(\theta)$ is given by

$$\gamma_*(\theta) = \begin{cases} \max(g(\theta), g(\pi/2 - \theta)) & \alpha < 1 \\ \min(\gamma_-(\theta), \gamma_+(\theta)) & \alpha \geq 1, \end{cases}$$

where $g(\theta) = |\cos\theta| \, (1 - |\tan\theta|^\alpha)^{1/\alpha}$. The following result shows that $\gamma_{\min}(\cdot)$ is less than or equal to $\gamma_*(\cdot)$.

Proposition 3 *For normalized components,* $0 \leq \gamma_{\min}(\theta) \leq \gamma_*(\theta)$.

Proof Consider the $\alpha < 1$ case. First assume θ_0 is in the interval $(0, \pi/4)$. For $\lambda_1 > 0$, define $\theta_1 = \theta_0 + \pi/2$ and $\theta_2 = \theta_2(\alpha, \theta_0, \lambda_1) = \pi/2 + \arctan[(1 - \lambda_1 |\cos\theta_0|^\alpha)/(1 - \lambda_1 |\sin\theta_0|^\alpha)]$, and $\lambda_2 = \lambda_2(\alpha, \theta_0, \lambda_1) = (1 - \lambda_1 |\sin\theta_0|^\alpha)/|\cos\theta_2|^\alpha$. When $0 < \lambda_1 < \lambda^* := 1/|\cos\theta_0|^\alpha$, calculation shows that $\lambda_2 > 0$, $\theta_1 < \theta_2 < \pi$,

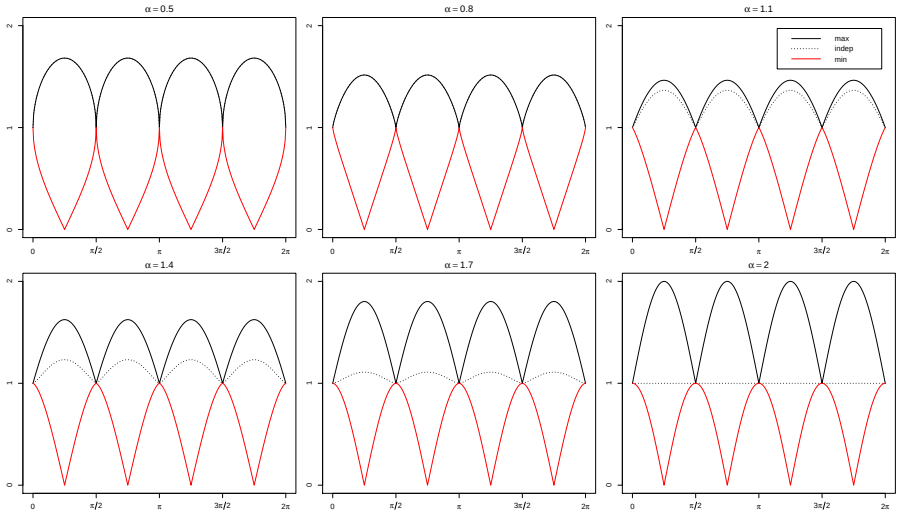

Fig. 3 Envelope functions $\gamma^{\alpha}_{\max}(\theta)$ (black) and conjectured $\gamma^{\alpha}_{\min}(\theta)$ (red) for different α. For comparison to the independent case, $\gamma^{\alpha}_{\perp}(\theta)$ is also shown as a dotted curve; when $\alpha \leq 1$, the independent scale function is identical to the upper bound.

and the bivariate stable distribution $\mathbf{X}_{\lambda_1}$ having the two point spectral measure with mass λ_j at θ_j has standardized components. For large λ_1, most of the mass is at θ_1, which is perpendicular to θ_0. The second point mass is placed to get normalized components. The corresponding scale function is $\gamma^{\alpha}_{\lambda_1}(\phi) = |\cos(\phi-\theta_1)|^{\alpha}\lambda_1 + |\cos(\phi-\theta_2)|^{\alpha}\lambda_2$. So $\gamma^{\alpha}_{\lambda_1}(\theta_0) = 0\cdot\lambda_1 + |\cos(\theta_0-\theta_2)|^{\alpha}\lambda_2$. As $\lambda_1 \uparrow \lambda*$, $\gamma^{\alpha}_{\lambda_1}(\theta_0) \downarrow g(\lambda^*)$.

For θ_0 in the interval $(\pi/4, \pi/2)$, the argument can be reflected to get the $g(\pi/2-\theta_0)$ bound. For the rest of the interval $(\pi/2, 2\pi)$, the argument can be shifted from the first quadrant. When $\alpha \geq 1$, $\gamma_*(\cdot)$ is achieved by $\gamma_-(\cdot)$ in the first and third quadrants, and by $\gamma_+(\cdot)$ in the second and fourth quadrants. □

For any scale function with normalized components, $\gamma(\theta) \leq \gamma_{\max}(\pi/4) = 2^{1/2}$ when $1 \leq \alpha \leq 2$, and $\gamma(\theta) \leq \gamma_{\max}(\pi/4) = 2^{1/\alpha-1/2}$ when $0 < \alpha \leq 1$. This latter term is unbounded as $\alpha \downarrow 0$. However $\gamma^{\alpha}(\theta) \leq 2$ for all α and all θ; this is why we used the α-th power in the definition of η_p. Figure 3 shows the α-th power of the upper envelope function $\gamma_{\max}(\cdot)$ and the proposed lower envelope $\gamma_*(\cdot)$.

Unlike the Pickands function, convexity is not necessary for $\gamma^{\alpha}(\cdot)$ to be a valid scale function, $\gamma^{\alpha}(\cdot)$ must be of positive type, or equivalently, $\exp(-\gamma^{\alpha}(\cdot))$ must be non-negative definite. We are not aware of any intrinsic characterization of a function $\gamma^{\alpha}(\cdot)$ that guarantees this.

Since $\gamma^{\alpha}(\cdot) \leq 2$ for any scale function with normalized components, η_p is always bounded by 4π, but this bound is not sharp. We conjecture that for any $0 < \alpha \leq 2$, any $p \geq 1$, and any normalized bivariate α-stable distribution (X_1, X_2), $0 \leq$

$\eta_p(X_1, X_2) \leq \eta_p^*$, where

$$\eta_p^* = \left\|\gamma_+^\alpha(\theta) - \gamma_\perp^\alpha(\theta)\right\|_p = \left\|\gamma_-^\alpha(\theta) - \gamma_\perp^\alpha(\theta)\right\|_p .$$

In words, the farthest distributions away from independence are the exact linear dependent cases. These values of η_p^* can be compute numerically.

We briefly compare η_p to the covariation and co-difference. For $\alpha > 1$, the covariation is defined by the first term and Example 2.7.3 in Samorodnitsky and Taqqu (1994) shows the second equality:

$$[X_1, X_2]_\alpha = \int_{\mathbb{S}} s_1 s_2^{<\alpha-1>} \Lambda(d\mathbf{s}) = \frac{1}{\alpha} \left.\frac{\partial \gamma^\alpha(u_1, u_2)}{\partial u_1}\right|_{(u_1=0, u_2=1)} .$$

Thus the covariation depends only on the behavior of $\gamma(\cdot, \cdot)$ near the point $(0, 1)$. If X_1 and X_2 are independent, then $[X_1, X_2]_\alpha = 0$. The discussion below shows that converse is false: there are many dependent distributions where $[X_1, X_2]_\alpha = 0$.

The co-difference is defined for symmetric α-stable vectors, and can be written as

$$\tau = \tau(X_1, X_2) = \gamma^\alpha(1, 0) + \gamma^\alpha(0, 1) - \gamma^\alpha(1, -1),$$

and is defined for any $\alpha \in (0, 2)$. It uses the values of the scale function at three particular points. If X_1 and X_2 are independent, then $\tau = 0$; when $\alpha \leq 1$, the converse is true, see Section 2.10 of Samorodnitsky and Taqqu (1994). When $\alpha > 1$, the converse is false, however if both $\tau(X_1, X_2) = 0$ and $\tau(X_1, -X_2) = 0$, then X_1 and X_2 are independent, see Section 2 of Rosiński and Zak (1997). A direct proof of this is straightforward: the condition $\tau(X_1, X_2) = 0$ is equivalent to

$$\int_{\mathbb{S}} |s_1|^\alpha \Lambda(d\mathbf{s}) + \int_{\mathbb{S}} |s_1|^\alpha \Lambda(d\mathbf{s}) = \int_{\mathbb{S}} |s_1 - s_2|^\alpha \Lambda(d\mathbf{s}),$$

and the condition $\tau(X_1, -X_2) = 0$ is equivalent to

$$\int_{\mathbb{S}} |s_1|^\alpha \Lambda(d\mathbf{s}) + \int_{\mathbb{S}} |s_1|^\alpha \Lambda(d\mathbf{s}) = \int_{\mathbb{S}} |s_1 + s_2|^\alpha \Lambda(d\mathbf{s}).$$

By Lemma 2.7.14 (2) in Samorodnitsky and Taqqu (1994), these two conditions together are equivalent to $s_1 s_2 = 0$ Λ-a.e., i.e. X_1 and X_2 are independent. As a consequence, one can define $\tau^2(X_1, X_2) + \tau^2(X_1, -X_2)$ as a measure that characterizes independence. However the sample analog of this has less power then the sample analog of η_2 proposed below.

Chapter 2 of Samorodnitsky and Taqqu (1994) gives properties of covariation and co-difference. Here we elaborate some on these properties, with attention to when they can be zero. Let Q_j be the open j-th quadrant, $j = 1, 2, 3, 4$.

For the covariation we restrict to $\alpha > 1$. Expressing the covariation in polar coordinates gives

$$[X_1, X_2]_\alpha = \int_0^{2\pi} \cos\theta (\sin\theta)^{<\alpha-1>} \Lambda(d\theta).$$

The integrand is zero precisely at the "poles" $\theta = 0, \pi/2, \pi$, and $3\pi/2$, which is precisely the support of any independent stable vector. So independence implies 0 covariation. But there are many other ways this can happen. The integrand function above is strictly positive on $Q_1 \cup Q_3$, and strictly negative on $Q_2 \cup Q_4$. Hence, if Λ is supported on $Q_1 \cup Q_3$, $[X_1, X_2]_\alpha > 0$; if Λ is supported on $Q_2 \cup Q_4$, $[X_1, X_2]_\alpha < 0$. Furthermore, if support Λ satisfies either condition, then normalizing the components of the corresponding distribution also has support in the corresponding region. So, there are many distributions with positive covariation and many with negative covariation, even if we restrict to normalized components.

The integral definition shows that covariation is linear in the spectral measure: for $c_0, c_1 \geq 0$, the covariation corresponding to $c_0\Lambda_0 + c_1\Lambda_1$ is the sum of c_0 times the covariation corresponding to Λ_0 plus c_1 times the covariation corresponding to Λ_1. Thus if Λ_0 has positive covariation and Λ_1 has negative covariation, setting $c_j = |\text{covariation of } \Lambda_{1-j}| \,/(|\text{covariation of } \Lambda_0| + |\text{covariation of } \Lambda_1|)$, $c_0\Lambda_0 + c_1\Lambda_1$ has covariation 0. Since this is a convex combination, if Λ_0 and Λ_1 have normalized components, so does the sum. Hence, there are many normalized dependent distributions with zero covariation.

For the co-difference, any $\alpha \in (0, 2]$ is allowed. Scaling shows $\tau = \gamma^\alpha(1, 0) + \gamma^\alpha(0, 1) - 2^{\alpha/2}\gamma^\alpha(1/\sqrt{2}, -1/\sqrt{2})$, so it suffices to consider $\gamma(\cdot)$ on the unit circle. Expressing the spectral measure in polar coordinates also gives a polar expression for τ:

$$\begin{aligned}\tau &= \gamma^\alpha(0) + \gamma^\alpha(\pi/2) - 2^{\alpha/2}\gamma^\alpha(3\pi/4) \\ &= \int_0^{2\pi} \left[|\cos\phi|^\alpha + |\sin\phi|^\alpha - 2^{\alpha/2}|\cos(3\pi/4 - \phi)|^\alpha \right] \Lambda(d\phi). \end{aligned} \tag{3}$$

The integrand above is 0 at the "poles", so independence implies 0 co-difference. Some special cases are straightforward to compute. In the elliptical case, $\gamma_{\text{ellip}}(\mathbf{u}) = 1 + 2\rho u_1 u_2$, so $\tau_{\text{ellip}} = 1 + 1 - (1 + 2\rho(-1)) = 2 - (2(1-\rho))^{\alpha/2}$. The range of τ_{ellip} is $[2 - 2^\alpha, 2]$; the lower bound is negative if and only if $\alpha > 1$. In the isotropic case, $\rho = 0$ and $\tau = 2 - 2^{\alpha/2}$. For $\gamma_+(\cdot)$, $\tau_+ = 1 + 1 - 2^{\alpha/2}|\cos\pi/2|^\alpha = 2$. This achieves max. For $\gamma_-(\cdot)$, $\tau_- = 1 + 1 - 2^{\alpha/2}2^{\alpha/2}|\cos 0|^\alpha = 2 - 2^\alpha$. This is positive for $\alpha < 1$, zero for $\alpha = 1$, and negative for $\alpha > 1$.

When $\alpha > 1$, the integrand function in (3) is strictly positive in $Q_1 \cup Q_3$ and strictly negative in $Q_2 \cup Q_4$. So, as with the covariation above, any spectral measure concentrated in $Q_1 \cup Q_3$ has strictly positive co-difference, and any spectral measure concentrated in $Q_2 \cup Q_4$ has strictly negative co-difference. As above, if Λ satisfies either of the conditions on the support, then normalizing the coordinates gives a spectral measure with support that satisfies the same condition.

We can also follow the argument above for covariation of sums to co-difference. Here we use notation τ_Λ for the co-difference for (X_1, X_2) having spectral measure Λ. It is simple to see that for spectral measures Λ_0 and Λ_1 and non-negative c_0, c_1, $\tau_{c_0\Lambda_0 + c_1\Lambda_1} = c_0\tau_{\Lambda_0} + c_1\tau_{\Lambda_1}$. In particular, if $\tau_{\Lambda_0} < 0$, $\tau_{\Lambda_1} > 0$ and $c_j = |\tau_{\Lambda_{1-j}}|/(|\tau_{\Lambda_0}| + |\tau_{\Lambda_1}|)$, then $\tau_{c_0\Lambda_0 + c_1\Lambda_1} = 0$. This shows that for a fixed α, the range of τ is an interval. Combined with the previous example, this gives many examples with $\alpha > 1$ where $\tau = 0$, but X_1 and X_2 are dependent. If Λ_0 and Λ_1 have unit scales, then so does the convex combination $c_0\Lambda_0 + c_1\Lambda_1$.

When $\alpha \leq 1$, the situation is different. In this case, the integrand in (3) is strictly positive off the "poles", so $\tau \geq 0$. The only way τ can be zero is therefore when the components are independent.

Combining the results above, we have shown that for a fixed α, $\min(0, 2-2^\alpha) \leq \tau \leq 2$. For $\alpha \geq 1$, all values in this region can be achieved by an elliptical stable law with some ρ. For $\alpha < 1$, all values can be achieved with spectral measures the sum of an independent and exact positive independence. Most values of τ can be achieved by many different distributions.

3 Measuring independence in a bivariate sample

Next we consider the statistical problem of determining the independence for a bivariate sample $\mathbf{X}_1, \ldots, \mathbf{X}_n$ of α-stable vectors. We start by normalizing the data: estimate the parameters $(\widehat{\alpha}_j, \widehat{\beta}_j, \widehat{\gamma}_j, \widehat{\delta}_j)$ of each component, $j = 1, 2$ using maximum likelihood as in Nolan (2001). Then use a pooled estimate of α: set $\widehat{\alpha} = (\widehat{\alpha}_1 + \widehat{\alpha}_2)/2$. To avoid numerical problems around $\alpha = 1$, we will only consider the 0-parameterization in this section. Then normalize the data $\mathbf{Y}_i = ((X_{i,1} - \delta_1)/\gamma_1, (X_{i,2} - \delta_2)/\gamma_2)$, $i = 1, \ldots, n$.

We note that $\gamma(-\mathbf{u}) = \gamma(\mathbf{u})$, so as in (2) it suffices to restrict the θ_j's to be in the interval $[0, \pi]$. For a set of angles $0 \leq \theta_1 < \theta_2 < \cdots < \theta_m \leq \pi$, define $\widehat{\gamma}_j = \widehat{\gamma}(\cos\theta_j, \sin\theta_j) =$ scale of the projected data set $\langle \mathbf{Y}_i, (\cos\theta_j, \sin\theta_j)\rangle$, $i = 1, \ldots, n$ and $\gamma_{\perp,j}^{\widehat{\alpha}} = |\cos\theta_j|^{\widehat{\alpha}} + |\sin\theta_j|^{\widehat{\alpha}}$, $j = 1, \ldots, m$. We will discuss the choice of grid below.

A straightforward sample approximation to $\eta_2(Y_1, Y_2)$ for the normalized data set is

$$\widehat{\eta}_2^2 = \widehat{\eta}_2^2(m, n) = \sum_{j=1}^{m} \left(\widehat{\gamma}_j^{\widehat{\alpha}} - \gamma_{\perp,j}^{\widehat{\alpha}}\right)^2. \tag{4}$$

We propose using this as a test statistic to determine the independence of a bivariate data set. The distribution of $\widehat{\eta}_2$ can be approximated using simulation.

For a formal test of independence, the null hypothesis is H_0: $\mathbf{X}$ has independent components. We compute the sample measure of independence $\widehat{\eta}_2$ and reject H_0 if $\widehat{\eta}_2 > c_{\epsilon,\alpha,\beta_1,\beta_2,n,m}$, where the ϵ is the desired significance level. We have written an R program to compute $\widehat{\eta}_2$ and estimate $c_{\epsilon,\alpha,\beta_1,\beta_2,n,m}$. Below we shows several examples.

We now discuss the choice of grid points $\theta_1, \ldots, \theta_m$. A uniform grid is not optimal: because of the normalizing, $\widehat{\gamma}(\mathbf{u}) = 1$ at $\mathbf{u} \in \{(1,0), (0,1), (-1,0), (0,-1)\}$ so we get no information including those points in the grid. Furthermore, when angles θ_j and θ_k are close, the corresponding directional scales are correlated. Figure 4 shows a plot of the covariance surface in one case. The discussion in Section 2 shows that the maximum distance between $\gamma(\theta)$ and $\gamma_\perp(\theta)$ can occur at the angles $\pi/4$ and $3\pi/4$. A reasonable choice is to pick an integer k and then define $\theta_j = j\pi/(2m)$, $j = 1, \ldots, k$ and $\theta_{j+k} = \theta_j + \pi/2$. This gives a grid of length $m = 2k$ that is uniformly spread on the interior of the first and third quadrant. Figure 1 shows the $k = 3$ case. The choice of k is complicated for several reasons. There is little information

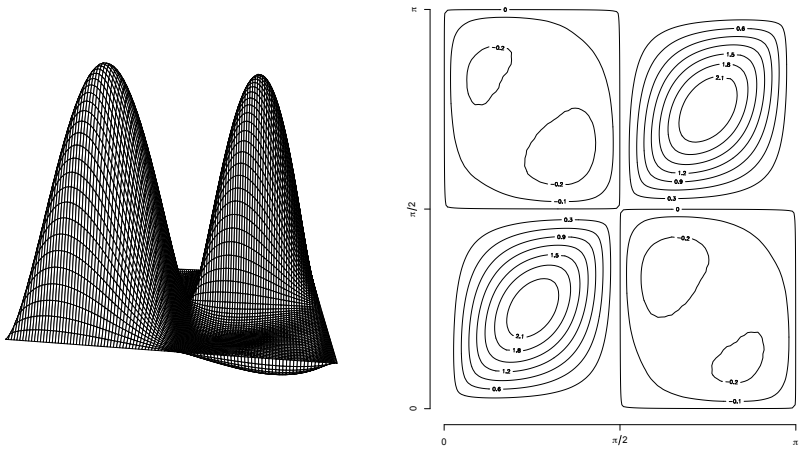

Fig. 4 Plot of the empirical covariance of $\gamma^{\alpha}(\cdot)$ for $\alpha = 1.5$, $\beta_1 = \beta_2 = 0$. The plot was generated by running $M = 20000$ simulations of sample size $n = 1000$ with independent components.

gained by getting close to the points (1,0), (0,1) and (-1,0) where the scale is fixed at 1. As k increases, the estimators of $\widehat{\gamma}_j$ and $\widehat{\gamma}_i$ are highly correlated. Unless n is very large, the variability in the estimators $\widehat{\gamma}_j$ will swamp the information gained by adding more grid points. We recommend $k = 5$ as a reasonable grid size for most practical applications. With very large sample sizes, there can be an increase in power by taking larger values of k.

An alternative approach is a χ^2 type of statistic. Let $\Sigma = \Sigma(n, \widehat{\alpha}, \beta_1, \beta_2) = [\sigma_{i,j}]_{i,j=1}^{m}$ be the covariance matrix of $\widehat{\gamma}_j^{\widehat{\alpha}}$, $j = 1, \ldots, m$ and let $\widehat{\boldsymbol{\gamma}}^{\widehat{\alpha}} = (\widehat{\gamma}_1^{\widehat{\alpha}}, \ldots, \widehat{\gamma}_m^{\widehat{\alpha}})^T$ and $\boldsymbol{\gamma}_{\perp}^{\widehat{\alpha}} = (\gamma_{\perp,1}^{\widehat{\alpha}}, \ldots, \gamma_{\perp,m}^{\widehat{\alpha}})^T$. Then define

$$\widehat{\eta}_{*,2}^2 = (\widehat{\boldsymbol{\gamma}}^{\widehat{\alpha}} - \boldsymbol{\gamma}_{\perp}^{\widehat{\alpha}})^T \, \Sigma^{-1} \, (\widehat{\boldsymbol{\gamma}}^{\widehat{\alpha}} - \boldsymbol{\gamma}_{\perp}^{\widehat{\alpha}}).$$

This is the square of the Mahalanobis distance when the data is from a stable distribution with indpendent components, and has $\chi^2(m)$ sampling distribution. This approach has the advantage of using tabulated critical values of the χ^2 distribution. However, we have been unable to prove that the vector $\widehat{\boldsymbol{\gamma}}$ is jointly normal or derive an analytic expression for $\Sigma = \Sigma(\alpha, \beta_1, \beta_2, \theta_1, \ldots, \theta_m)$.

Next we analyze some financial data. Closing price data on two pharmaceutical stocks, Pfizer (symbol PFE) and Merck (MRK), was gathered for the five year time period January 1, 2010 to December 31, 2014 resulting in 1257 prices. For each stock, log returns were computed and stable parameters were estimated using maximum likelihood for each company. For Pfizer, $\widehat{\alpha} = 1.748$, $\widehat{\beta} = 0.0000$, $\widehat{\gamma} = 0.0070$ and $\widehat{\delta} = 0.00585$; for Merck, $\widehat{\alpha} = 1.735$, $\widehat{\beta} = -0.0852$, $\widehat{\gamma} = 0.0070$ and $\widehat{\delta} = 0.000684$. The indices are close, so we used $\alpha = (1.748 + 1.735)/2 = 1.7415$ and computed

sample size n	isotropic	independent $\circlearrowleft \pi/4$	independent $\circlearrowleft \pi/8$	independent $\circlearrowleft \pi/16$	exact linear dependence
25	0.191	0.322	0.243	0.213	1
50	0.223	0.624	0.381	0.183	1
100	0.344	0.918	0.644	0.214	1
200	0.636	0.998	0.937	0.440	1
300	0.874	1	0.997	0.627	1
400	0.960	1	1	0.791	1
500	0.989	1	1	0.893	1
600	0.999	1	1	0.959	1
700	1	1	1	0.980	1
800	1	1	1	0.985	1
900	1	1	1	0.998	1
1000	1	1	1	0.997	1

Table 1 Power of the statistic $\widehat{\eta}_2$ for various types of dependence when $\alpha = 1.5$.

the test statistic $\widehat{\eta}_2 = 1.3381$. Simulations with 1256 independent stable terms shows that this value is highly significant: the critical value for $p = 0.01$ is 0.241, so we reject the null hypothesis that the terms are independent. This is not surprising as PFE and MRK are in the same sector. We repeated this procedure with PFE and Walmart (WMT), which has a similar α and also reject independence at the p=0.01 level. In fact, we computed $\widehat{\eta}_2$ for every pair of stocks in the Dow Jones 30 index, and all reject independence at this level. These results should be interpreted cautiously, as estimates of α varied between 1.62 and 1.87 for different stocks and there were $C(30, 2) = 435$ comparisons made.

Table 1 shows estimates of the power for detecting dependence for several types of dependence. The isotropic column is when the data is radially symmetric, the third, fourth and fifth columns are counterclockwise rotations of the independent spectral measure by the stated angle, the last column shows exact linear positive dependence (all the mass of the spectral measure on the diagonal line $y = x$). The isotropic case is a modest departure from independence, and shows to reliably detect this kind of dependence, same sizes in the several hundreds are required. The third column shows that when the spectral measure is concentrated on the diagonal and anti-diagonal lines, $\widehat{\eta}_2$ reliably detects dependence with sample sizes on the order of 100 or more. As the rotation lessens, the corresponding distribution is closer and closer to independence, and larger samples are needed to reliably detect dependence. The last column shows that quite small samples of size $n = 25$ are sufficient to detect dependence when the data is concentrated on the diagonal line.

3.1 Higher dimensions

For $\mathbf{X} = (X_1, \ldots, X_d)$, $d > 2$, we can apply this method to each pair of coordinates and assess the d-dimensional data set. Specifically, for each pair of indices we compute $\eta_{i,j} = \eta_2(X_i, X_j)$. Recall from above that the d dimensional data set is independent if and only if each pair of components is independent. Perhaps more im-

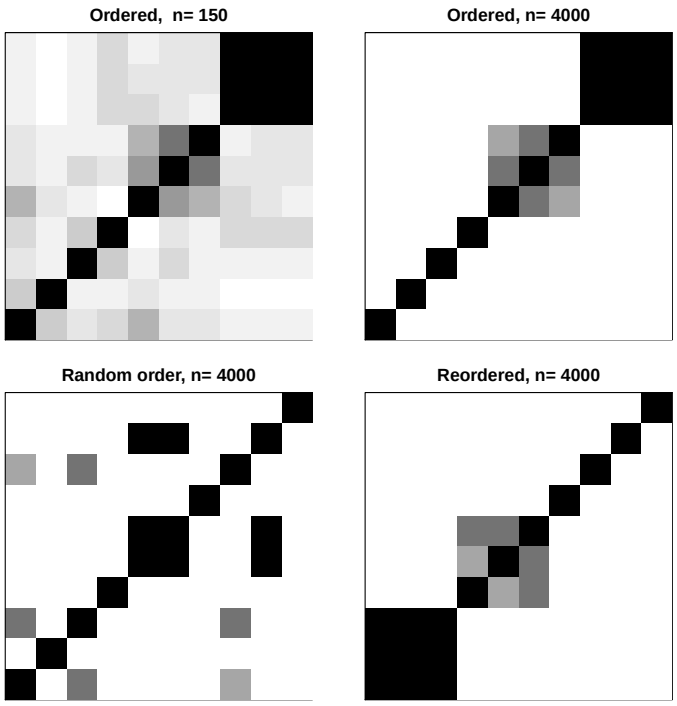

Fig. 5 Plot of $\widehat{\eta}_2$ for a simulated 10 dimensional data set with $\alpha = 1.3$, and dependence structure as described in the text.

portant is the view of the joint dependence structure in a multivariate data set given by the η_2 matrix.

One application of this idea is in dimension reduction. If the η matrix shows obvious structure, e.g. blocks of dependence, then one can split the d dimensional modeling problem into two or more lower dimensional problems. This is illustrated in Figure 5, which shows a greyscale plot of the pairwise values of η_2 for a simulated 10 dimensional data set that is $\alpha = 1.3$ stable. The first four coordinates of the simulated vector are independent, the next three are from an elliptical stable distribution with shape matrix $R = (1, 0.5, 0.25; 0.5, 1, 0.5; 0.25, 0.5, 1)$, and the last three are from a discrete spectral measure with mass spread around the first octant. The blocks are independent of each other. The upper left plot shows the estimated $\widehat{\eta}_2$ matrix for a modest sample size of $n = 150$, where the dependence structure is visible, but not yet sharp. In upper right plot, the sample size is increased to $n = 4000$, and now the dependence structure is sharp. In the bottom left plot, the coordinates are randomly permuted to hide the dependence structure. Finally, the bottom right plot shows the reordered (from the permuted data) $\widehat{\eta}_2$ matrix, with the ordering coming from a clustering algorithm. (Specifically, we used the R function `hclust(dist(·))` applied to the $\widehat{\eta}_2$ matrix.) This approach may be useful in applications to discover structure in a heavy tailed multivariate data set.

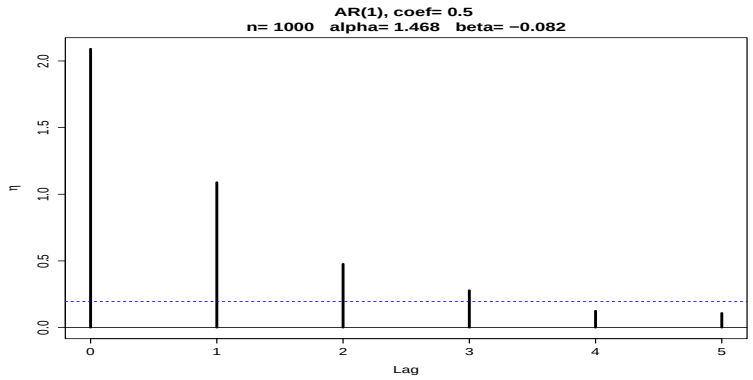

Fig. 6 $\eta_2(X_i, X_{i+j})$ for a simulated AR(1) time series. The horizontal line is the critical value for rejecting independence when $n = 1000$, $\alpha = 1.5$, $\beta_1 = \beta_2 = 0$, and $\epsilon = 0.05$.

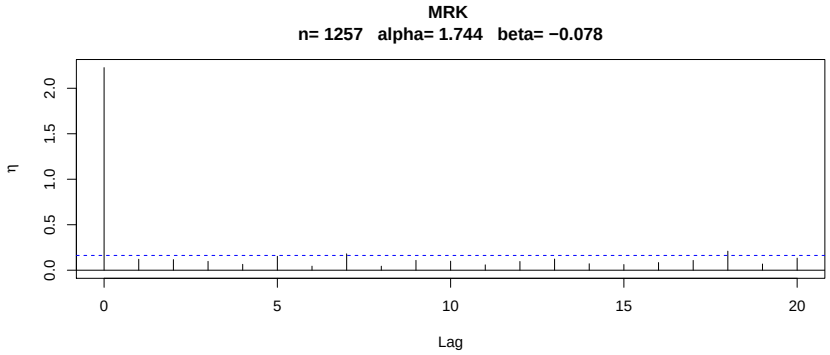

Fig. 7 $\eta_2(X_i, X_{i+h})$ for Merck returns for $\epsilon = 0.05$.

3.2 Application to times series

Given a univariate time series $X_1, X_2, \ldots, X_n$ stable error terms, the above definition of independence can be used to define an analogy of the autocorrelation function. The approach is similar: compute the dependence measure $\eta_{2,h}$ for lagged pairs (X_{t+h}, X_t), $t = 1, \ldots, n-h$. Plot $\eta_{2,h}$ as a function of h and show a threshold value as in the ACF plot.

Figure 6 shows an example of this with simulated data. In this case, and AR process is simulated: $X_t = cX_{t-1} + Z_t$, where $c = 1/2$ and Z_t are normalized i.i.d. symmetric α-stable.

This method is applied to the financial returns of Merck stock that was examined above. Figure 7 shows that there is no evidence for dependence among the lagged values of the returns.

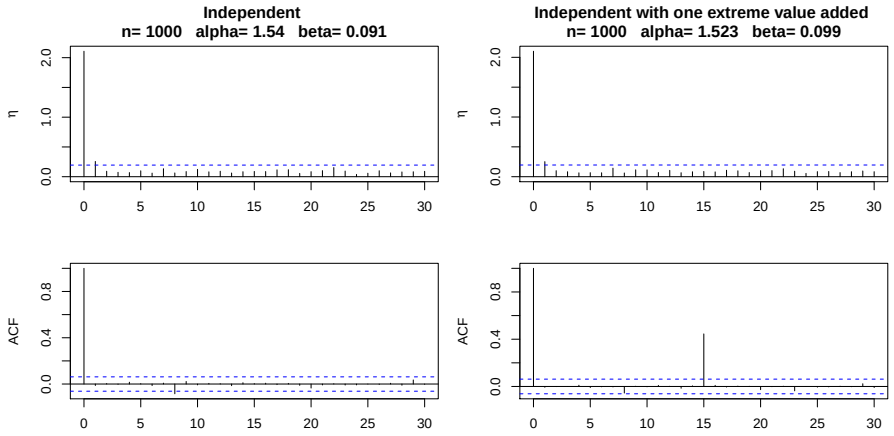

Fig. 8 $\eta_2(X_i, X_{i+j})$ for a simulated stable time series with $\epsilon = 0.05$, $\alpha = 1.5$, and independent terms. The top left shows a plot of $\widehat{\eta}_2$ for the original time series, the bottom left shows the standard acf plot for the same data. On the right side, one point in the time series was changed.

One advantage of η over the standard ACF is robustness. While generally the η plot and the acf plot look similar, the latter is sensitive to extreme values. This is illustrated in Figure 8, where a time series with independent terms is analyzed on the left side of the plot using both the η plot and a standard acf plot. Then one point is changed in the time series in the following way: we looked for the maximum value in the time series (which was 204.62 in this simulation), and replaced a value 15 time periods away with 80% of this maximum. The η plot and the standard acf were then graphed for this altered time series on the right set of plots. Changing this one point makes acf plot shows a strong spike at lag 15, whereas the η plot does not. Thus one changed value, significantly changes the ACF, but not the η plot.

3.3 Vectors in the domain of attraction of a stable law

In the preceding sections we assumed the bivariate vector was stable. We now show how the method can be adapted to vectors in the domain of attraction of a stable law. We will use the notation $\mathbf{X} \in \text{DOA}(\mathbf{S}(\alpha, \beta(\cdot), \gamma(\cdot), \delta(\cdot); 0))$ when $\mathbf{X}$ is in the domain of attraction of a $\mathbf{S}(\alpha, \beta(\cdot), \gamma(\cdot), \delta(\cdot); 0)$ law. Our approach is straightforward: use any univariate tail estimator for $\widehat{\alpha}$ and the scale function and then use the above approach for stable.

There are a range of methods to choose here; we use the simplest to illustrate the approach. Sort the data and look at $\log x$ vs. the log of $1 - \widehat{F}(x)$, the complement of the empirical distribution function, beyond some threshold. A simple linear regression will give an estimate of the tail index and the scale. A similar approach can be done for the lower tail. For simplicity, we will assume that the data is two sided, and we average the tail indices to get $\widehat{\alpha}$ and scale $\widehat{\gamma}$.

sample size n	isotropic	independent $\circlearrowleft \pi/4$	independent $\circlearrowleft \pi/8$	independent $\circlearrowleft \pi/16$	exact linear dependence
100	0.253	0.057	0.049	0.058	0.161
200	0.708	0.025	0.040	0.049	0.342
300	0.844	0.010	0.013	0.023	0.481
400	0.940	0.011	0.020	0.022	0.995
500	0.956	0.011	0.007	0.018	1
600	0.986	0.024	0.013	0.028	1
700	0.988	0.023	0.003	0.009	1
800	0.995	0.258	0.012	0.019	1
900	0.998	0.284	0.013	0.011	1
1000	0.993	0.498	0.006	0.009	1
2000	1	0.996	0.376	0.008	1
3000	1	1	0.876	0.003	1
4000	1	1	0.989	0.003	1
5000	1	1	1	0.004	1

Table 2 Power of the statistic $\widehat{\eta}_2$ for $\mathbf{X}$ in the domain of attraction of an $\alpha = 1.5$ stable law, see the text for data description.

Equipped with a one dimensional estimate of the tail index and scale, we proceed as we did in the exact stable case: (i) analyze the marginals and average the resulting tail indices to get an estimate of α; (ii) pick a grid of directions $\theta_1, \ldots, \theta_m$; (iii) project the data along each direction, getting estimates of the scale $\widehat{\gamma}_1, \ldots, \widehat{\gamma}_m$; (iv) compute $\widehat{\eta}_2$ using (4).

To compute a critical value, we use non-parametric bootstrapping: generate M data sets, each time generating independent vectors, with each component sampled with replacement from the sample component; compute $\widehat{\eta}_2$ for this bootstrap sample; tabulate the values of $\widehat{\eta}_2$ and find the appropriate quantile.

To test this method, we reproduced Table 1 for the domain of attraction case. We generated several data sets based on Pareto terms: we simulated $\mathbf{X} = (X_1 - X_1', X_2 - X_2')$, where each term is independent Pareto($\alpha = 1.5$). This is a symmetric r. vector with independent components that is in the domain of attraction of an independent $\alpha = 1.5$ stable law. For each n, this was simulated $M = 10000$ times and $\widehat{\eta}_2$ was calculated using the tail estimator modification. In this and the following examples, $k = 5$ grid points were used in each quadrant for a total of $m = 2k = 10$ grid points and 10% of the tails were used to estimate the tail index and scale of both the lower and the upper tails. From these simulatinos, the 0.95-percentile was calculated and used as critical values were used for each of the cases considered in Table 2. For each column in that table we simulated $M = 1000$ data sets of the specified type and computed the fraction of times that independence is rejected at the 95% level. For the column labeled isotropic, we generated $R \sim$ Pareto($\alpha = 1.5$) and $\theta \sim$ Uniform$(0, 2\pi)$ and set $\mathbf{X} = (R\cos\theta, R\sin\theta)$. For the next three columns, we generated independent terms as in the critical value calculations and then rotated those vectors by the specified angles. For the exact linear dependence, we simulated $\mathbf{X} = \epsilon(R, R)$, where $\epsilon \pm 1$ with probability 1/2 and $R \sim$ Pareto($\alpha = 1.5$).

This table starts at higher values of n then the earlier table because we could not get consistent results with smaller sample sizes. Recall that the tail estimator is based

on the bottom and top 10% of a sample, and without sample sizes in the hundreds, little can be done. The power values are much lower than the ones in the exact stable case because most of the data is not being used by the tail estimator. We note that these results are dependent on multiple factors: the tail estimation method, the fraction of the tail used to estimate the index and scale, the marginal distributions of the data, etc. Finally, we note that the this method will be unreliable if the threshold is not chosen well or the available data doesn't show the limiting behavior clearly.

4 Compliance with Ethical Standards

Funding: as disclosed on the title page, Nolan was supported by an agreement with Cornell University, Operations Research & Information Engineering under W911NF-12-1-0385 from the Army Research Development and Engineering Command. Alparslan does not have funding at the current time.

Research involving Human Participants and/or Animals: not applicable.

Informed consent: Prof. Alparslan and Nolan are the only people involved in this work, both participated in the submission of this manuscript.

Conflict of Interest: The authors declare that they have no conflict of interest.

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