HIDDEN REGULAR VARIATION OF MOVING AVERAGE PROCESSES WITH HEAVY-TAILED INNOVATIONS

SIDNEY I. RESNICK AND JOYJIT ROY

Abstract. We look at joint regular variation properties of MA(∞) processes of the form $\mathbf{X} = (X_k, k \in \mathbb{Z})$ where $X_k = \sum_{j=0}^{\infty} \psi_j Z_{k-j}$ and the sequence of random variables $(Z_i, i \in \mathbb{Z})$ are iid with regularly varying tails. We use the setup of $\mathbb{M}_{\mathbb{O}}$-convergence and obtain hidden regular variation properties for $\mathbf{X}$ under summability conditions on the constant coefficients $(\psi_j : j \geq 0)$. Our approach emphasizes continuity properties of mappings and produces regular variation in sequence space.

1. Introduction

The purpose of this paper is to obtain joint regular variation properties of moving average processes of the form

$$X_k = \sum_{j=0}^{\infty} \psi_j Z_{k-j}, \ k \in \mathbb{Z},$$

where Z_i are iid nonnegative heavy-tailed random variables and ψ_j are constant nonnegative coefficients. The study of tail behavior of such processes has a long history. Early studies of the one-dimensional case with constant coefficients are [22, 4, 8, 23]; see also accounts in [20, 3]. The d-dimensional results as well as results for moving average processes with random coefficients can be found in [21, 12, 16]. Joint regular variation properties of the MA(∞) process were obtained in [8]. Many of these studies emphasized finding proper summability conditions for the coefficient sequence which forces the extremal properties of the process to be determined by the tail behavior of the innovation sequence. In this paper we use a fairly strong summability assumption on the coefficient sequence and concentrate on using continuity arguments to obtain joint regular variation properties of the entire sequence as a random element of the space of double-sided sequences.

Traditionally multivariate regular variation properties of d-dimensional random vectors have been expressed using vague convergence of measures where limit measures are finite on compact sets. To make extremal sets which contain neighborhoods of infinities and are unbounded above compact, the approach has been to compactify a locally compact space such as $[0,\infty)^2$ by adding lines through ∞ to obtain $[0,\infty]^2$ and then restrict the class of sets on which the limit measure has to be finite by removing a point such as $(0,0)$ to obtain $[0,\infty]^2 \setminus \{(0,0)\}$. See

2010 *Mathematics Subject Classification.* 28A33,60G70,37M10).

Key words and phrases. regular variation, multivariate heavy tails, hidden regular variation, moving average processes.

S. Resnick and J. Roy were supported by Army MURI grant W911NF-12-1-0385 to Cornell University.

[13, 17, 19]. There are systemic problems inherent to using vague convergence theory such as dealing with lines through ∞ and points of uncompactification when addressing continuous mapping arguments. Further the theory is limited to locally compact spaces.

An alternate framework for dealing with tail behavior of general random elements, the theory of $\mathbb{M}_{\mathbb{O}}$-convergence was developed in [10] which applies the theory of $w_{\#}$-convergence ([6]) to obtain a framework which lends itself nicely to dealing with regular variation on any complete, separable metric space with a point removed. The theory was further extended to allow consideration of spaces with a general closed cone removed in [14]. The main attraction of any such theory lies in powerful mapping theorems and their use to obtain results about transformations and functionals (see [9, 11]). In this paper we prove that $\mathbf{X} = (X_k, k \in \mathbb{Z})$ is regularly varying as an element of $\mathbb{R}^{\infty}_{+,\mathbb{Z}} \setminus \{\mathbf{0}_\infty\}$ using such a mapping argument.

Another aspect of multivariate regular variation that is relevant to our paper is the concept of hidden regular variation which was first developed in [18, 15]. As a simple example, we can look at 2 concurrent regular variation properties of an iid Pareto(1) pair of random variables (X_1, X_2). Observe that for $x_1, x_2 \geq 0$,

$$tP[(X_1 > tx_1, X_2 > tx_2] \to \begin{cases} (x_1 \vee x_2)^{-1} & \text{if} \qquad x_1 \wedge x_2 = 0 \\ 0 & \text{o.w.} \end{cases}$$

while

$$tP[(X_1 > t^{1/2}x_1, X_2 > t^{1/2}x_2] \to (x_1x_2)^{-1} \text{ if } x_1 \wedge x_2 > 0.$$

Here the second regular variation property of the pair (X_1, X_2) is only applicable on a part of the state space obtained by removing the support of the limit measure in the first regular variation property and then using a scaling function that goes to ∞ slower than t. The second property was hidden by the coarse scaling used to obtain convergence to a non-zero measure in the first case. The theory of $\mathbb{M}_{\mathbb{O}}$-convergence has already been fruitfully applied to prove the existence of hidden regular variation ([7]). In this paper we obtain an infinite sequence of hidden regular variation properties for the finite moving average process as an element of $\mathbb{R}^{\infty}_{+,\mathbb{Z}}$.

In Section 2, we define $\mathbb{M}_{\mathbb{O}}$-convergence and collect relevant results about the theory as well as the definition of regular variation of a random variable in this framework. In Section 3.1, we restate results about regular variation of iid heavy-tailed sequences obtained in [14] which will form the basis for proving our results. In Section 3.3 we prove the existence of hidden regular variation for the MA(m) process before proving our main theorem in Section 3.4. Due to technical considerations, proving a hidden regular variation property for the MA(∞) sequence has not yet been possible but the authors are working towards achieving that end; we instead prove hidden regular variation for finite order moving averages. Still our main result about the joint regular variation of the entire sequence not only serves as a nice demonstration of the power of continuous mapping theorems in the $\mathbb{M}_{\mathbb{O}}$ framework but will serve as a building block for obtaining further results through the use of other mappings and functionals to the sequence space.

2. Basics of $\mathbb{M}_{\mathbb{O}}$-convergence and regular variation of measures

In this section we define the framework for $\mathbb{M}_{\mathbb{O}}$-convergence and collect basic results that will be useful later. For more details and proofs see Sections 2 and 3 in [14].

2.1. $\mathbb{M}_{\mathbb{O}}$-convergence.

Let $(\mathbb{S}, d)$ be a complete separable metric space with Borel σ-algebra $\mathscr{S}$ generated by open sets. Fix a closed set $\mathbb{C} \subset \mathbb{S}$ and set $\mathbb{O} = \mathbb{S} \setminus \mathbb{C}$, The subspace $\mathbb{O}$ is a metric subspace of $\mathbb{S}$ in the relative topology with σ-algebra $\mathscr{S}(\mathbb{O}) = \{A : A \subset \mathbb{O}, A \in \mathscr{S}\}$.

Let $\mathscr{C}_b$ denote the class of real-valued, non-negative, bounded and continuous functions on $\mathbb{S}$, and let $\mathbb{M}_b$ denote the class of finite Borel measures on $\mathscr{S}$. A basic neighborhood of $\mu \in \mathbb{M}_b$ is a set of the form $\{\nu \in \mathbb{M}_b : |\int f_i \, \mathrm{d}\nu - \int f_i \, \mathrm{d}\mu| < \varepsilon, i = 1, \dots, k\}$, where $\varepsilon > 0$ and $f_i \in \mathscr{C}_b$ for $i = 1, \dots, k$. This equips $\mathbb{M}_b$ with the weak topology and convergence $\mu_n \to \mu$ in $\mathbb{M}_b$ means $\int f \mathrm{d}\mu_n \to \int f \mathrm{d}\mu$ for all $f \in \mathscr{C}_b$. See, for example, Sections 2 and 6 in [1] for details.

Let $\mathscr{C}(\mathbb{O})$ denote the real-valued, non-negative, bounded and continuous functions f on $\mathbb{O}$ such that for each f there exists $r > 0$ such that f vanishes on $\mathbb{C}^r$; we use the notation $\mathbb{C}^r = \{x \in \mathbb{S} : d(x, \mathbb{C}) < r\}$, where $d(x, \mathbb{C}) = \inf_{y \in \mathbb{C}} d(x, y)$. Similarly, we will write $d(A, \mathbb{C}) = \inf_{x \in A,\, y \in \mathbb{C}} d(x, y)$ for $A \subset \mathbb{S}$. We say that a set $A \in \mathscr{S}(\mathbb{O})$ is bounded away from $\mathbb{C}$ if $A \subset \mathbb{S} \setminus \mathbb{C}^r$ for some $r > 0$ or equivalently $d(A, \mathbb{C}) > 0$. So $\mathscr{C}(\mathbb{O})$ consists of non-negative continuous functions whose supports are bounded away from $\mathbb{C}$. Let $\mathbb{M}_{\mathbb{O}}$ be the class of Borel measures on $\mathbb{O} = \mathbb{S} \setminus \mathbb{C}$, whose restriction to $\mathbb{S} \setminus \mathbb{C}^r$ is finite for each $r > 0$. When convenient, we also write $\mathbb{M}(\mathbb{O})$ or $\mathbb{M}(\mathbb{S} \setminus \mathbb{C})$. A basic neighborhood of $\mu \in \mathbb{M}_{\mathbb{O}}$ is a set of the form $\{\nu \in \mathbb{M}_{\mathbb{O}} : |\int f_i \mathrm{d}\nu - \int f_i \mathrm{d}\mu| < \varepsilon,\ i = 1, \dots, k\}$, where $\varepsilon > 0$ and $f_i \in \mathscr{C}(\mathbb{O})$ for $i = 1, \dots, k$. Convergence $\mu_n \to \mu$ in $\mathbb{M}_{\mathbb{O}}$ is convergence in the topology defined by this base. As the next theorem shows ([14, Theorem 2.1]), it actually suffices to consider the class of uniformly continuous functions in $\mathscr{C}(\mathbb{O})$.

Theorem 2.1. *Let $\mu, \mu_n \in \mathbb{M}_{\mathbb{O}}$. Then the following statements are equivalent.*
(i) $\mu_n \to \mu$ in $\mathbb{M}_{\mathbb{O}}$ as $n \to \infty$.
(ii) $\int f \mathrm{d}\mu_n \to \int f \mathrm{d}\mu$ for each $f \in \mathscr{C}(\mathbb{O})$ which is also uniformly continuous on $\mathbb{S}$.
(iii) $\mu_n^{(r)} \to \mu^{(r)}$ in $\mathbb{M}_b(\mathbb{S} \setminus \mathbb{C}^r)$ for all $r > 0$ such that $\mu(\partial \mathbb{S} \setminus \mathbb{C}^r) = 0$ where $\mu^{(r)}$ denote the restriction of μ to $\mathbb{S} \setminus \mathbb{C}^r$.

Continuous mapping theorems will play an important role in extending the regular variation property of the innovation sequence to that of the actual moving average sequence. Here we state one version that will prove useful to us. Consider another separable and complete metric space $\mathbb{S}'$ and let $\mathbb{O}', \mathscr{S}_{\mathbb{O}'}, \mathbb{C}', \mathbb{M}_{\mathbb{O}'}$ have the same meaning relative to the space $\mathbb{S}'$ as do $\mathbb{O}, \mathscr{S}_{\mathbb{O}}, \mathbb{C}, \mathbb{M}_{\mathbb{O}}$ relative to $\mathbb{S}$.

Theorem 2.2. *Suppose $h : \mathbb{S} \mapsto \mathbb{S}'$ is uniformly continuous and $\mathbb{C}' := h(\mathbb{C})$ is closed in $\mathbb{S}'$. Then $\hat{h} : \mathbb{M}_{\mathbb{O}} \mapsto \mathbb{M}_{\mathbb{O}'}$ defined by $\hat{h}(\mu) = \mu \circ h^{-1}$ is continuous.*

2.2. Regular variation of measures.

The usual notion of regular variation involves comparisons along a ray and requires a concept of scaling or multiplication. Given any real number $\lambda > 0$ and any $x \in \mathbb{S}$, we assume there exists a mapping $(\lambda, x) \mapsto \lambda x$ from $(0, \infty) \times \mathbb{S}$ into $\mathbb{S}$ satisfying:

- the mapping $(\lambda, x) \mapsto \lambda x$ is continuous,

- $1x = x$ and $\lambda_1(\lambda_2 x) = (\lambda_1\lambda_2)x$.

These two assumptions allow definition of a cone $\mathbb{C} \subset \mathbb{S}$ as a set satisfying $x \in \mathbb{C}$ implies $\lambda x \in \mathbb{C}$ for any $\lambda > 0$. For this section, fix a closed cone $\mathbb{C} \subset \mathbb{S}$ and then $\mathbb{O} := \mathbb{S} \setminus \mathbb{C}$ is an open cone. Also assume that

- $d(x, \mathbb{C}) < d(\lambda x, \mathbb{C})$ if $\lambda > 1$ and $x \in \mathbb{O}$.

Recall from, for example, [2] that a positive measurable function c defined on $(0, \infty)$ is regularly varying with index $\rho \in \mathbb{R}$ if $\lim_{t\to\infty} c(\lambda t)/c(t) = \lambda^\rho$ for all $\lambda > 0$. Similarly, a sequence $\{c_n\}_{n\geq 1}$ of positive numbers is regularly varying with index $\rho \in \mathbb{R}$ if $\lim_{n\to\infty} c_{[\lambda n]}/c_n = \lambda^\rho$ for all $\lambda > 0$. Here $[\lambda n]$ denotes the integer part of λn.

Definition 2.3. *A sequence $\{\nu_n\}_{n\geq 1}$ in $\mathbb{M}_{\mathbb{O}}$ is regularly varying if there exists an increasing sequence $\{c_n\}_{n\geq 1}$ of positive numbers which is regularly varying and a nonzero $\mu \in \mathbb{M}_{\mathbb{O}}$ such that $c_n\nu_n \to \mu$ in $\mathbb{M}_{\mathbb{O}}$ as $n \to \infty$.*

We now define regular variation for a single measure in $\mathbb{M}_{\mathbb{O}}$ as well as an equivalent formulation that is more pleasing to handle algebraically.

Definition 2.4. *A measure $\nu \in \mathbb{M}_{\mathbb{O}}$ is regularly varying if the sequence $\{\nu(n\cdot)\}_{n\geq 1}$ in $\mathbb{M}_{\mathbb{O}}$ is regularly varying or equivalently there exist a nonzero $\mu \in \mathbb{M}_{\mathbb{O}}$ and an increasing function b such that $t\nu(b(t)\cdot) \to \mu(\cdot)$ in $\mathbb{M}_{\mathbb{O}}$ as $t \to \infty$. Similarly we say that a random variable $\mathbf{Y}$ taking values in $\mathbb{S}$ is regularly varying if the associated probability measure is regularly varying, i.e. if $P[\mathbf{Y} \in \cdot]$ is regularly varying.*

We will refer to the function b as the scaling function corresponding to the regularly varying measure ν on $\mathbb{M}_{\mathbb{O}}$.

3. Main results

3.1. Hidden regular variation for iid heavy tailed sequences.

From here on we will look at $\mathbb{S} = \mathbb{R}^\infty_{+,\mathbb{Z}}$ where $\mathbb{R}^\infty_{+,\mathbb{Z}}$ is defined to be the space of all double-sided sequences of non-negative real numbers i.e. $\mathbb{R}^\infty_{+,\mathbb{Z}} = \{\mathbf{x} = (x_i, i \in \mathbb{Z}) : x_i \geq 0\}$ equipped with the metric $d_{\infty,\mathbb{Z}}$ defined as

$$d_{\infty,\mathbb{Z}}(\mathbf{x}, \mathbf{y}) = \sum_{i=-\infty}^{\infty} \frac{|x_i - y_i| \wedge 1}{2^{|i|+1}}. \tag{3.1}$$

The concept of multiplication will be given by the standard pointwise multiplication of a sequence by a real number.

Observe that convergence in this metric is equivalent to convergence of all finite dimensional sequences, i.e. $d_{\infty,\mathbb{Z}}(\mathbf{x}^{\mathbf{n}}, \mathbf{x}) \to 0$, if and only if for any $M \in \mathbb{Z}_+$, the sequences $(x_i^n, |i| \leq M)$ converge pointwise to $(x_i, |i| \leq M)$ in $\mathbb{R}^{2M+1}$. Further observe that $(\mathbb{R}^\infty_{+,\mathbb{Z}}, d_{\infty,\mathbb{Z}})$ as a metric space is homeomorphic to $(\mathbb{R}^\infty_+, d_\infty)$ where $\mathbb{R}^\infty_+ = \{\mathbf{x} = (x_i, i \in \mathbb{N}) : x_i \geq 0\}$ and $d_\infty(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^\infty \frac{|x_i - y_i| \wedge 1}{2^i}$.

Define $\mathbf{0}_\infty$ to be the sequence with all components 0 in $\mathbb{R}^\infty_{+,\mathbb{Z}}$ and $\mathbf{e}_i$ to be the sequence in $\mathbb{R}^\infty_{+,\mathbb{Z}}$ with the ith component 1 and all other components 0. Further define

$$\begin{aligned}\mathbb{C}_{=j} &= \{\mathbf{x} \in \mathbb{R}^{\infty}_{+,\mathbb{Z}} : \sum_{i=-\infty}^{\infty} \boldsymbol{\epsilon}_{x_i}((0,\infty)) = j\} \text{ for all } j \geq 1, \text{ and} \\ \mathbb{C}_{\leq j} &= \{\mathbf{x} \in \mathbb{R}^{\infty}_{+,\mathbb{Z}} : \sum_{i=-\infty}^{\infty} \boldsymbol{\epsilon}_{x_i}((0,\infty)) \leq j\} \text{ for all } j \geq 0.\end{aligned} \tag{3.2}$$

Define $\mathbb{O}_j = \mathbb{R}^{\infty}_{+,\mathbb{Z}} \setminus \mathbb{C}_{\leq j-1}$ for $j \geq 1$.

Let $\mathbf{Z} = (Z_i, i \in \mathbb{Z})$ be iid random variables in $\mathbb{R}_+$ with regularly varying tails with index $\alpha > 0$ i.e.

$$\lim_{t\to\infty} \frac{P[Z_0 > tz]}{P[Z_0 > t]} = z^{-\alpha} \text{ for all } z > 0. \tag{3.3}$$

or equivalently for some regularly varying sequence $b(\cdot)$,

$$\lim_{t\to\infty} tP[Z_0 > b(t)z] = z^{-\alpha} \text{ for all } z > 0. \tag{3.4}$$

With this setup, we can restate Theorem 4.2 in [14] as a statement about the space $\mathbb{R}^{\infty}_{+,\mathbb{Z}}$ and the sequence of iid random variables $\mathbf{Z} \in \mathbb{R}^{\infty}_{+,\mathbb{Z}}$. Define for each $j \geq 0$,

$$\begin{aligned}\mu_t^{(j)}(\cdot) &= tP(\mathbf{Z}/b(t^{1/(j+1)}) \in \cdot) \text{ and} \\ \mu^{(j)}(\cdot) &= \sum_{(i_1,\dots,i_{j+1})} \int \mathbb{1}\left\{\sum_{k=1}^{j+1} z_k \mathbf{e}_{i_k} \in \cdot\right\} \nu_\alpha(\mathrm{d}z_1)\dots\nu_\alpha(\mathrm{d}z_{j+1}),\end{aligned}$$

where $\nu_\alpha(x,\infty) = x^{-\alpha}$ and the indices $(i_1,\dots,i_{j+1})$ run through the ordered subsets of size $j+1$ of $\mathbb{Z}$.

Theorem 3.1. *For every $j \geq 0$, $\mu_t^{(j)} \to \mu^{(j)}$ in $\mathbb{M}(\mathbb{O}_j)$. The measure $\mu^{(j)}$ concentrates on $\mathbb{C}_{\leq j+1} \setminus \mathbb{C}_{\leq j} = \mathbb{C}_{=j+1}$ and has the alternative form*

$$\begin{aligned}&\mu^{(j)}(\dots,\mathrm{d}z_1,\mathrm{d}z_0,\mathrm{d}z_{-1},\dots) \\ &\quad= \sum_{(i_1,\dots,i_{j+1})} \left(\prod_{k\notin\{i_1,\dots,i_{j+1}\}} \boldsymbol{\epsilon}_0(\mathrm{d}z_k)\right)\left(\prod_{k\in\{i_1,\dots,i_{j+1}\}} \nu_\alpha(\mathrm{d}z_k)\right).\end{aligned} \tag{3.5}$$

3.2. **Definition of the MA(∞) process and framework of proof of the main result.** Let $(\psi_j, j \geq 0)$ be a sequence of non-negative constants with

- (A1) $\psi_0 > 0$ and
- (A2) for some $\delta < \alpha \wedge 1$, $\sum_{j=0}^{\infty} \psi_j^{\delta} < \infty$.

Observe that Assumption (A2) implies the following:

- (C1) $\sum_{j=0}^{\infty} \psi_j < \infty$,
- (C2) $\sum_{j=0}^{\infty} \psi_j^{\alpha} < \infty$,
- (C3) for any $k \in \mathbb{Z}$, the sequence $\sum_{j=0}^{\infty} \psi_j Z_{k-j}$ converges almost surely, and
- (C4) for any $x > 0$ and $k \in \mathbb{Z}$,

$$\lim_{N\to\infty} \limsup_{t\to\infty} tP\left[\sum_{j>N} \psi_j Z_{k-j} > b(t)x\right] = 0,$$

where $\mathbf{Z} = (Z_i, i \in \mathbb{Z})$ is defined in (3.4). (C1) and (C2) are easy to see while proofs of (C3) and (C4) can be found in [17] (Section 4.5, especially Lemma 4.24) and [4, 5].

Remark. Assumption (A2) is not the weakest condition known in the literature that implies (C1 - C4). See [24, 12, 21, 16] for different summability assumptions on the sequence $(\psi_j, j \geq 0)$ as well as a treatment of moving average processes with random coefficients and heavy tailed innovations.

For $k \in \mathbb{Z}$, define

$$X_k = \sum_{j=0}^{\infty} \psi_j Z_{k-j}.$$

(C3) ensures that $\mathbf{X} = (X_k, k \in \mathbb{Z})$ is a well defined sequence of random variables in $\mathbb{R}^{\infty}_{+,\mathbb{Z}}$. If we define the map $\mathbf{T}^{\infty} : \mathbb{R}^{\infty}_{+,\mathbb{Z}} \ni \mathbf{z} = (z_i, i \in \mathbb{Z}) \mapsto \mathbf{T}^{\infty}(\mathbf{z}) = (T^{\infty}_k(\mathbf{z}), k \in \mathbb{Z}) \in \mathbb{R}^{\infty}_{+,\mathbb{Z}}$ where

$$T^{\infty}_k(\mathbf{z}) = \sum_{j=0}^{\infty} \psi_j z_{k-j}, \tag{3.6}$$

then we have that

$$\mathbf{X} = \mathbf{T}^{\infty}(\mathbf{Z}).$$

This leads us to suspect that regular variation properties can be obtained from Theorem 3.1 using a continuous mapping argument. But unfortunately, the map $\mathbf{T}^{\infty}$, even though well-defined P-almost surely, is nowhere continuous on $\mathbb{R}^{\infty}_{+,\mathbb{Z}}$. This forces us to use a truncation argument as detailed in the sequel by using a sequence of maps which map $\mathbf{Z}$ to the partial sums of the infinite sums that make up the elements of $\mathbf{X}$ and then using a Slutsky-style approximation. The details are technical as we are dealing with infinite measures.

3.3. **Hidden regular variation of the MA(m) process.** For every $m \geq 0$, define the random variable $\mathbf{X}^m = (X^m_k, k \in \mathbb{Z}) \in \mathbb{R}^{\infty}_{+,\mathbb{Z}}$ where

$$X^m_k = \sum_{j=0}^{m} \psi_j Z_{k-j}.$$

Similar to (3.6) we can define for every $m \geq 0$ the map $\mathbf{T}^m : \mathbb{R}^{\infty}_{+,\mathbb{Z}} \ni \mathbf{z} = (z_i, i \in \mathbb{Z}) \mapsto \mathbf{T}^m(\mathbf{z}) = (T^m_k(\mathbf{z}), k \in \mathbb{Z}) \in \mathbb{R}^{\infty}_{+,\mathbb{Z}}$ where

$$T^m_k(\mathbf{z}) = \sum_{j=0}^{m} \psi_j z_{k-j}.$$

Again we have that $\mathbf{X}^m = \mathbf{T}^m(\mathbf{Z})$. However the map $\mathbf{T}^m$ is well-behaved enough for us to use Theorem 2.2. We first prove two preliminary lemmas to enable the use of said theorem, which will lead to our main result about the MA(m) processes.

Lemma 3.2. *For every $m \geq 0$, the map $\mathbf{T}^m$ is uniformly continuous.*

Proof. Fix $m \geq 0$ and $\epsilon > 0$. Let $M > 0$ be such that $2.2^{-M} < \varepsilon/2$. Take $\mathbf{x} = (x_i, i \in \mathbb{Z}), \mathbf{y} = (y_i, i \in \mathbb{Z}) \in \mathbb{R}^\infty_{+,\mathbb{Z}}$. Then

$$d_{\infty,\mathbb{Z}}(\mathbf{T}^m(\mathbf{x}), \mathbf{T}^m(\mathbf{y})) < \sum_{|i|<M} \frac{|\sum_{j=0}^m \psi_j x_{i-j} - \sum_{j=0}^m \psi_j y_{i-j}| \wedge 1}{2^{|i|+1}} + \varepsilon/2$$

$$\leq 2(\sum_{j=0}^m \psi_j)(\bigvee_{|i|<M+m} |x_i - y_i|) + \varepsilon/2 \tag{3.7}$$

Let $\delta < (\sum_{j=0}^m \psi_j)\frac{\varepsilon}{4}2^{-(M+m)}$ and assume that $d_{\infty,\mathbb{Z}}(\mathbf{x}, \mathbf{y}) < \delta$. Then from (3.1), we get that

$$\bigvee_{|i|<M+m} |x_i - y_i| < 2^{(M+m)}\delta < (\sum_{j=0}^m \psi_j)\frac{\varepsilon}{4}.$$

Then using (3.7) we get that

$$d_{\infty,\mathbb{Z}}(\mathbf{T}^m(\mathbf{x}), \mathbf{T}^m(\mathbf{y})) < \varepsilon.$$

□

Lemma 3.3. *For every $m \geq 0$ and $j \geq 0$, $\mathbf{T}^m(\mathbb{C}_{\leq j})$ is closed, where $\mathbb{C}_{\leq j}$ is defined as in* (3.2).

Proof. Fix $m \geq 0$ and observe that for $j = 0$, $\mathbf{T}^m(\mathbb{C}_{\leq 0}) = \mathbf{T}^m(\{\mathbf{0}_\infty\}) = \{\mathbf{0}_\infty\}$ which is trivially closed. This settles the base case for a proof of the result by induction. Now we proceed to prove the result by induction. So assume that the result holds for $j < J$. Take $\mathbf{z^n} \in \mathbb{C}_{=J}$ such that $\mathbf{z}^n \to \mathbf{z} \in \mathbb{R}^\infty_{+,\mathbb{Z}}$. It is enough to assume this as $\mathbb{C}_{\leq J} = \mathbb{C}_{\leq J-1} \cup \mathbb{C}_{=J}$. Further assume that we have sequences $\lambda_1^n, \lambda_2^n, \ldots, \lambda_J^n > 0$ and $i_1^n < i_2^n < \ldots < i_J^n \in \mathbb{Z}$ such that

$$\mathbf{z}^n = \sum_{k=1}^J \lambda_k^n \mathbf{e}_{i_k^n} \text{ and}$$
$$\mathbf{T}^m(\mathbf{z}^n) = \sum_{l=0}^m \sum_{k=1}^J \psi_l \lambda_k^n \mathbf{e}_{i_k^n - j}.$$

Observe that if $i_J^n \to -\infty$ along some subsequence n_q, then the limit of any finite dimensional subsequence of $\mathbf{T}^m(\mathbf{z}^{n_q})$ is the same as the finite dimensional subsequential limit of $\mathbf{T}^m(\sum_{k=1}^{J-1} \lambda_k^{n_q} \mathbf{e}_{i_k^{n_q}})$. Since the limit of a sequence in $\mathbb{R}^\infty_{+,\mathbb{Z}}$ is determined by the limits of the finite dimensional subsequences, we have by the induction hypothesis that $\mathbf{z} \in \mathbf{T}^m(\mathbb{C}_{\leq J-1}) \subset \mathbf{T}^m(\mathbb{C}_{\leq J})$. A similar argument shows that if $i_1^n \to \infty$ along some subsequence then $\mathbf{z} \in \mathbf{T}^m(\mathbb{C}_{\leq J})$. So we must have that $(i_1^n, i_2^n, \ldots, i_J^n)$ are contained in some bounded set and so they must equal some $(i_1, i_2, \ldots, i_J)$ infinitely often where $i_1 < i_2 < \ldots < i_J$. Without loss of generality we may now assume that

$$\mathbf{z}^n = \sum_{k=1}^J \lambda_k^n \mathbf{e}_{i_k} \text{ and}$$
$$\mathbf{T}^m(\mathbf{z}^n) = \sum_{l=0}^m \sum_{k=1}^J \psi_l \lambda_k^n \mathbf{e}_{i_k - j}.$$

Since $(\mathbf{T}^m(\mathbf{z}^n))_{i_J-m} = \psi_0\lambda_J^n$ converges and by assumption (A1), $\psi_0 > 0$, we must have that $\lambda_J^n \to \lambda$ for some $\lambda \geq 0$. This leads to $\mathbf{T}^m(\sum_{k=1}^{J-1}\lambda_k^n \mathbf{e}_{i_k}) \to \mathbf{z} - \mathbf{T}^m(\lambda\mathbf{e}_{i_J})$. But the induction hypothesis now gives us that $\mathbf{z} - \mathbf{T}^m(\lambda\mathbf{e}_{i_J}) \in \mathbf{T}^m(\mathbb{C}_{\leq J-1})$. Thus we get that $\mathbf{z} \in \mathbf{T}^m(\mathbb{C}_{\leq J})$ proving the induction step. $\square$

A quick application of Theorem 2.2 now gives us the following result.

Theorem 3.4. *For every $m \geq 0$ and $j \geq 0$, $\mu_t^{(j)} \circ (\mathbf{T}^m)^{-1} \to \mu^{(j)} \circ (\mathbf{T}^m)^{-1}$ in $\mathbb{M}(\mathbb{R}^\infty_{+,\mathbb{Z}} \setminus \mathbf{T}^m(\mathbb{C}_{\leq j}))$ or equivalently*

$$tP\left[\mathbf{X}^m/b(t^{1/(j+1)}) \in \cdot\right] \to \sum_{(i_1,\ldots,i_{j+1})} \int \mathbb{1}\left\{\mathbf{T}^m(\sum_{k=1}^{j+1} z_k \mathbf{e}_{i_k}) \in \cdot\right\} \nu_\alpha(\mathrm{d}z_1)\ldots\nu_\alpha(\mathrm{d}z_{j+1}).$$

Remark. Observe that Theorem 3.4 implies an infinitude of regular variation properties for $\mathbf{X}^m$.

For example, for $j = 0$,

$$tP\left[\mathbf{X}^m/b(t) \in \cdot\right] \to \nu^{m,(0)}(\cdot) \text{ in } \mathbb{M}(\mathbb{R}^\infty_{+,\mathbb{Z}} \setminus \mathbf{T}^m(\{\mathbf{0}_\infty\})) = \mathbb{M}(\mathbb{R}^\infty_{+,\mathbb{Z}} \setminus \{\mathbf{0}_\infty\}),$$

where

$$\nu^{m,(0)}(\cdot) = \sum_{i=-\infty}^{\infty} \int \mathbb{1}\left\{\mathbf{T}^m(z_i\mathbf{e}_i) \in \cdot\right\} \nu_\alpha(\mathrm{d}z_i).$$

From the above it is clear that $\nu^{m,(0)}$ is a non-zero measure, it is finite on subsets of $\mathbb{R}^\infty_{+,\mathbb{Z}}$ bounded away from $\mathbf{0}_\infty$ and its support is on $\mathbf{T}^m(\mathbb{C}_{=1})$. Thus $\mathbf{X}^m$ is regularly varying on $\mathbb{R}^\infty_{+,\mathbb{Z}} \setminus \{\mathbf{0}_\infty\}$ with scaling function $b(\cdot)$ and limit measure $\nu^{m,(0)}$. Using (3.5) assuming $\psi_j > 0$ for all $j \leq m$, we have the following alternate and slightly more illuminating formulation for $\nu^{m,(0)}$, namely

$$\begin{aligned}&\nu^{m,(0)}(\ldots,\mathrm{d}z_1,\mathrm{d}z_0,\mathrm{d}z_{-1},\ldots)\\ &\quad= \sum_{i=-\infty}^{\infty}\left(\prod_{k<i}\epsilon_0(\mathrm{d}z_k)\right)\left(\prod_{i\leq k\leq i+m}\nu_\alpha\left(\frac{\mathrm{d}z_k}{\psi_{k-i}}\right)\right)\left(\prod_{k>i+m}\boldsymbol{\epsilon}_0(\mathrm{d}z_k)\right).\end{aligned}$$

Further for any $k \in \mathbb{Z}$, we have that

$$tP[X_k^m > b(t)x] \to \left(\sum_{l=0}^{m}\psi_l^\alpha\right)x^{-\alpha} \text{ for } x > 0.$$

Similarly for $j = 1$,

$$tP\left[\mathbf{X}^m/b(t^{1/2}) \in \cdot\right] \to \nu^{m,(1)}(\cdot) \text{ in } \mathbb{M}(\mathbb{R}^\infty_{+,\mathbb{Z}} \setminus \mathbf{T}^m(\mathbb{C}_{\leq 1})),$$

where $\nu^{m,(1)}$ is a non-zero measure on $\mathbb{R}^\infty_{+,\mathbb{Z}} \setminus \mathbf{T}^m(\mathbb{C}_{\leq 1})$ with support $\mathbf{T}(\mathbb{C}_{=2})$. So $\mathbf{X}^m$ is also regularly varying on $\mathbb{R}^\infty_{+,\mathbb{Z}} \setminus \mathbf{T}^m(\mathbb{C}_{\leq 1})$ with scaling function $b(t^{1/2})$. Observe that for $j = 0$ we removed just $\mathbf{T}^m(\mathbb{C}_{\leq 0}) = \{\mathbf{0}_\infty\}$ from $\mathbb{R}^\infty_{+,\mathbb{Z}}$ and obtained that $\mathbf{X}^m$ was regularly varying with a limit measure concentrating on $\mathbf{T}^m(\mathbb{C}_{=1})$ which is a very small part of the entire state space $\mathbb{R}^\infty_{+,\mathbb{Z}} \setminus \{\mathbf{0}_\infty\}$. Now, on also removing the support of $\nu^{m,(0)}$ i.e. $\mathbf{T}^m(\mathbb{C}_{=1})$ from the state space we obtained a new regular variation property for $\mathbf{X}^m$ on a smaller state space $\mathbb{R}^\infty_{+,\mathbb{Z}} \setminus \mathbf{T}^m(\mathbb{C}_{\leq 1})$ with a finer scaling function $b(t^{1/2})$. This regular variation property was in some sense hidden by the cruder scaling that we used for the larger state space. This is a typical example of hidden regular variation. For a more expository account on such a nested sequence of regular variation properties in the case of iid heavy tailed random variables, see [14] (Section 4.5).

In fact we have an increasing sequence of cones,

$$\mathbf{T}^m(\mathbb{C}_{\le 0}) \subset \mathbf{T}^m(\mathbb{C}_{\le 1}) \subset \dots \mathbf{T}^m(\mathbb{C}_{\le j}) \subset \dots,$$

a sequence of non-zero measures $\nu^{m,(j)}, j \ge 0$ where $\nu^{m,(j)}$ is supported on $\mathbf{T}^m(\mathbb{C}_{\le j+1}) \setminus \mathbf{T}^m(\mathbb{C}_{\le j})$ and a sequence of decreasing scaling functions

$$b(t) > b(t^{1/2}) > \dots b(t^{1/(j+1)}) \dots .$$

such that $\mathbf{X}^m$ is regularly varying on $\mathbb{R}^{\infty}_{+,\mathbb{Z}} \setminus \mathbf{T}^m(\mathbb{C}_{\le j})$ with limit measure $\nu^{m,(j)}$ and scaling function $b(t^{1/(j+1)})$. Thus by removing more and more of the state space and using finer and finer scaling functions we are able to get a more detailed picture of the extremal properties of $\mathbf{X}^m$.

3.4. **Regular variation of the MA(∞) process.** As mentioned before the map $\mathbf{T}^\infty$ is only well-defined P-almost surely. For each $j > 0$, $\mathbf{T}^\infty(\mathbb{C}_{\le j})$ is not closed, even though $\mathbf{T}^\infty$ is well-defined on each $\mathbb{C}_{\le j}$. This prevents us from proving a result implying hidden regular variation of $\mathbf{X}$ as in Theorem 3.4 for $\mathbf{X}^m$. However, the fact that $\mathbf{T}^\infty(\{\mathbf{0}_\infty\}) = \{\mathbf{0}_\infty\}$ and the use of (C4) and interpreting $\mathbf{X}$ as the limit of $\mathbf{X}^m$ as $m \to \infty$, allows us to prove the following result. The proof, except for technical details, is similar in spirit to [19] (Theorem 3.5) or [1] (Theorem 3.2).

Theorem 3.5. $\mu_t^{(0)} \circ (\mathbf{T}^\infty)^{-1} \to \mu^{(0)} \circ (\mathbf{T}^\infty)^{-1} = \nu^{(0)}$ *in* $\mathbb{M}(\mathbb{R}^{\infty}_{+,\mathbb{Z}} \setminus \{\mathbf{0}_\infty\}) = \mathbb{M}(\mathbb{O}_0)$ *or equivalently*

$$tP\left[\mathbf{X}/b(t) \in \cdot\right] \to \sum_{i=\infty}^{\infty} \int \mathbb{1}\left\{\mathbf{T}^\infty(z_i \mathbf{e}_i) \in \cdot\right\} \nu_\alpha(\mathrm{d}z_i).$$

Remark. (i) Theorem 3.5 implies that $\mathbf{X}$ is regularly varying on $\mathbb{R}^{\infty}_{+,\mathbb{Z}} \setminus \mathbf{0}_\infty$ with limit measure $\nu^{(0)}$ and scaling function $b(\cdot)$. The limit measure $\nu^{(0)}$ can also be expressed in the following way emphasizing the fact that its support is on $\mathbf{T}^\infty(\mathbb{C}_{=1})$ and it is indeed non-zero,

$$\begin{aligned}&\nu^{(0)}(\dots, \mathrm{d}z_1, \mathrm{d}z_0, \mathrm{d}z_{-1}, \dots)\\ &\quad= \sum_{i=-\infty}^{\infty} \left(\prod_{k<i \text{ or } \psi_{k-i}=0} \boldsymbol{\epsilon}_0(\mathrm{d}z_k) \right) \left(\prod_{k \ge i \text{ and } \psi_{k-i}>0} \nu_\alpha\left(\frac{\mathrm{d}z_k}{\psi_{k-i}}\right) \right).\end{aligned}$$

Also for any $k \in \mathbb{Z}$, we have that

$$tP[X_k > b(t)x] \to \left(\sum_{l=0}^{\infty} \psi_l^\alpha\right) x^{-\alpha} \text{ for } x > 0.$$

(ii) It is also instructive to compare Theorem 3.5 to Theorem 2.4 in [8] where a point process version of the same result was obtained.
(iii) An application of the continuous mapping theorem (Theorem 2.2) allows us to prove regular variation for sums of MA(∞) processes from Theorem 3.5. For every $m \ge 0$, define the random variable $\mathbf{Y}^m = (Y_k^m, k \in \mathbb{Z}) \in \mathbb{R}^{\infty}_{+,\mathbb{Z}}$ where

$$Y_k^m = \sum_{j=0}^{m} X_{k-j}.$$

Observe that $\mathbf{Y}^m = \mathbf{SUM}^m(\mathbf{X})$ where, for every $m \geq 0$, the map $\mathbf{SUM}^m : \mathbb{R}^\infty_{+,\mathbb{Z}} \ni \mathbf{x} = (x_i, i \in \mathbb{Z}) \mapsto \mathbf{SUM}^m(\mathbf{x}) = (\sum_{j=0}^m x_{k-j}, k \in \mathbb{Z}) \in \mathbb{R}^\infty_{+,\mathbb{Z}}$. $\mathbf{SUM}^m$ is uniformly continuous by Lemma 3.2 and so we can apply Theorem 2.2 to $\nu^{(0)}$, to obtain that, in $\mathbb{M}(\mathbb{R}^\infty_{+,\mathbb{Z}} \setminus \{\mathbf{0}_\infty\})$,

$$tP\left[\mathbf{Y}^m/b(t) \in \cdot\right] \to \nu^{(0)} \circ (\mathbf{SUM}^m)^{-1}(\cdot).$$

Application of Theorem 3.5 to obtain regular variation of other functionals of MA(∞), such as sample covariances, is not so straightforward in the sense that proving uniform continuity of the corresponding function, such as $\mathbf{SUM}^m$ in the case of the summation functional, is non-trivial.

Before proving Theorem 3.5 we prove two technical lemmas. Set $\sum_{j=0}^\infty \psi_j = S$ which is finite by (C1).

Lemma 3.6. *For any $\gamma > 0$,*

$$\lim_{n\to\infty} \limsup_{t\to\infty} \mu_t^{(0)}(\{\mathbf{z} : d_{\infty,\mathbb{Z}}(\mathbf{T}^\infty(\mathbf{z}), \mathbf{0}_\infty) > \gamma, d_{\infty,\mathbb{Z}}(\mathbf{z}, \mathbf{0}_\infty) < \delta_n\}) = 0$$

where $\delta_n = 2^{-(n+1)}/S$.

Proof. Fix $\gamma > 0$ and let $M > 0$ be such that $2\sum_{|i|\geq M} 2^{-(|i|+1)} < \gamma/2$. Then we have

$$\begin{aligned}
&(\{\mathbf{z} : d_{\infty,\mathbb{Z}}(\mathbf{T}^\infty(\mathbf{z}), \mathbf{0}_\infty) > \gamma, d_{\infty,\mathbb{Z}}(\mathbf{z}, \mathbf{0}_\infty) < \delta_n\} \\
&\subset \left\{\mathbf{z} : \sum_{|i|<M} \frac{\mathbf{T}_i^\infty(\mathbf{z}) \wedge 1}{2^{|i|+1}} > \gamma/2, \sum_{|i|<n} \frac{\mathbf{z}_i \wedge 1}{2^{|i|+1}} < \delta_n\right\} \\
&\subset \left\{\mathbf{z} : \bigvee_{|i|<M} \mathbf{T}_i^\infty(\mathbf{z}) > \gamma, \bigvee_{|i|<n} z_i < 2^n\delta_n\right\} \\
&\subset \bigcup_{|i|<M} \left\{\mathbf{z} : \sum_{l=0}^\infty \psi_l z_{i-l} > \gamma, \bigvee_{|i|<n} z_i < 2^n\delta_n\right\} \\
&\subset \bigcup_{|i|<M} \left\{\mathbf{z} : (\sum_{l=0}^\infty \psi_l)(\bigvee_{|i|<n} z_i) + \sum_{l>i+n} \psi_l z_{i-l} > \gamma, \bigvee_{|i|<n} z_i < 2^n\delta_n\right\} \\
&\subset \bigcup_{|i|<M} \left\{\mathbf{z} : \sum_{l>i+n} \psi_l z_{i-l} > \gamma - S2^n\delta_n\right\} \\
&\subset \bigcup_{|i|<M} \left\{\mathbf{z} : \sum_{l>i+n} \psi_l z_{i-l} > \gamma/2\right\}
\end{aligned}$$

for large enough n. So we have that

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{t\to\infty} \mu_t^{(0)}(\{\mathbf{z} : d_{\infty,\mathbb{Z}}(\mathbf{T}^\infty(\mathbf{z}), \mathbf{0}_\infty) > \gamma, d_{\infty,\mathbb{Z}}(\mathbf{z}, \mathbf{0}_\infty) < \delta_n\}) \\
&\leq \sum_{|i|<M} \lim_{n\to\infty} \limsup_{t\to\infty} \mu_t^{(0)}\left(\left\{\mathbf{z} : \sum_{l>i+n} \psi_l z_{i-l} > \gamma/2\right\}\right)
\end{aligned}$$

$$\leq \sum_{|i|<M} \lim_{n\to\infty} \limsup_{t\to\infty} tP\left[\left\{\mathbf{z} : \sum_{l>i+n} \psi_l Z_{i-l} > b(t)\gamma/2\right\}\right]$$
$$= 0.$$

The last line follows from (C4). □

Lemma 3.7. *For any $\beta > 0$,*

$$\lim_{m\to\infty} \limsup_{t\to\infty} \mu_t^{(0)}(\{\mathbf{z} : d_{\infty,\mathbb{Z}}(\mathbf{T}^\infty(\mathbf{z}), \mathbf{T}^m(\mathbf{z})) > \beta\}) = 0$$

Proof. Fix $\beta > 0$ and let $M > 0$ be such that $2\sum_{|i|\geq M} 2^{-(|i|+1)} < \beta/2$. Then we have

$$\begin{aligned}
&\{\mathbf{z} : d_{\infty,\mathbb{Z}}(\mathbf{T}^\infty(\mathbf{z}), \mathbf{T}^m(\mathbf{z})) > \beta\}\\
&\subset \left\{\mathbf{z} : \sum_{|i|<M} \frac{|\mathbf{T}^\infty(\mathbf{z}) - \mathbf{T}^m(\mathbf{z})|_i \wedge 1}{2^{|i|+1}} > \beta/2\right\}\\
&\subset \left\{\mathbf{z} : \sum_{|i|<M} \frac{\sum_{l>m+1} \psi_l z_{i-l} \wedge 1}{2^{|i|+1}} > \beta/2\right\}\\
&\subset \left\{\mathbf{z} : \bigvee_{|i|<M} \sum_{l>m+1} \psi_l z_{i-l} > \beta\right\}\\
&\subset \bigcup_{|i|<M} \left\{\mathbf{z} : \sum_{l>m+1} \psi_l z_{i-l} > \beta\right\}.
\end{aligned}$$

As in Lemma 3.6, we have

$$\begin{aligned}
&\lim_{m\to\infty} \limsup_{t\to\infty} \mu_t^{(0)}(\{\mathbf{z} : d_{\infty,\mathbb{Z}}(\mathbf{T}^\infty(\mathbf{z}), \mathbf{T}^m(\mathbf{z})) > \beta\})\\
&\leq \sum_{|i|<M} tP\left[\left\{\mathbf{z} : \sum_{l>m+1} \psi_l Z_{i-l} > b(t)\beta\right\}\right],
\end{aligned}$$

which is 0 by (C4). □

Proof of Theorem 3.5. By Theorem 2.1, it is enough to show that for any uniformly continuous $f \in \mathscr{C}(\mathbb{O}_0)$, we have that $\int f\, \mathrm{d}\mu_t^{(j)} \circ (\mathbf{T}^\infty)^{-1} \to \int f\, \mathrm{d}\mu^{(j)} \circ (\mathbf{T}^\infty)^{-1}$. Fix any such f and set $\mathbb{F} = \{\mathbf{z} \in \mathbb{R}^\infty_{+,\mathbb{Z}} : f(\mathbf{z}) > 0\}$. Since $f \in \mathscr{C}(\mathbb{O}_0)$, we may assume that $d_{\infty,\mathbb{Z}}(\mathbb{F}, \mathbf{0}_\infty) > \gamma > 0$ and $\sup_{\mathbf{z}\in\mathbb{R}^\infty_{+,\mathbb{Z}}} f(\mathbf{z}) = 1$. Let $\omega_f(\cdot)$ be the modulus of continuity of f.

Fix $\varepsilon > 0$. By Lemma 3.6 we can find $\mathbb{G}$ such that $\mathbb{G} = \{\mathbf{z} \in \mathbb{R}^\infty_{+,\mathbb{Z}} : d_{\infty,\mathbb{Z}}(\mathbf{z}, \mathbf{0}_\infty) > \delta\}$ for some $\delta > 0$, $\mu^{(0)}(\partial\, \mathbb{G}) = 0$ and $\lim_{t\to\infty} \mu_t^{(0)}(\mathbb{F} \setminus \mathbf{T}^\infty(\mathbb{G})) = \mu^{(0)}(\mathbb{F} \setminus \mathbf{T}^\infty(\mathbb{G})) < \varepsilon$. Then

$$\begin{aligned}
&\left|\int f\, \mathrm{d}\mu_t^{(0)} \circ (\mathbf{T}^\infty)^{-1} - \int f\, \mathrm{d}\mu^{(0)} \circ (\mathbf{T}^\infty)^{-1}\right|\\
&= \left|\int_{\mathbf{T}^\infty(\mathbf{z})\in\mathbb{F}} f \circ \mathbf{T}^\infty(\mathbf{z})\ \mu_t^{(0)}(\mathrm{d}\mathbf{z}) - \int_{\mathbf{T}^\infty(\mathbf{z})\in\mathbb{F}} f \circ \mathbf{T}^\infty(\mathbf{z})\ \mu^{(0)}(\mathrm{d}\mathbf{z})\right|
\end{aligned}$$

$$\leq \left| \int_{\mathbb{G}} f \circ \mathbf{T}^{\infty}(\mathbf{z})\, \mu_t^{(0)}(\mathrm{d}\mathbf{z}) - \int_{\mathbf{G}} f \circ \mathbf{T}^{\infty}(\mathbf{z})\, \mu^{(0)}(\mathrm{d}\mathbf{z}) \right| \\ + \mu_t^{(0)}(\mathbb{F} \setminus \mathbf{T}^{\infty}(\mathbb{G})) + \mu^{(0)}(\mathbb{F} \setminus \mathbf{T}^{\infty}(\mathbb{G})).$$

Since the last two terms are less than ε for large enough t, it suffices to show that the first term in the last line above goes to 0.

$$\begin{aligned} &\left| \int_{\mathbb{G}} f \circ \mathbf{T}^{\infty}(\mathbf{z})\, \mu_t^{(0)}(\mathrm{d}\mathbf{z}) - \int_{\mathbf{G}} f \circ \mathbf{T}^{\infty}(\mathbf{z})\, \mu^{(0)}(\mathrm{d}\mathbf{z}) \right| \\ &\quad \leq \left| \int_{\mathbb{G}} f \circ \mathbf{T}^{\infty}(\mathbf{z})\, \mu_t^{(0)}(\mathrm{d}\mathbf{z}) - \int_{\mathbf{G}} f \circ \mathbf{T}^{m}(\mathbf{z})\, \mu_t^{(0)}(\mathrm{d}\mathbf{z}) \right| \\ &\quad\quad + \left| \int_{\mathbb{G}} f \circ \mathbf{T}^{m}(\mathbf{z})\, \mu_t^{(0)}(\mathrm{d}\mathbf{z}) - \int_{\mathbf{G}} f \circ \mathbf{T}^{m}(\mathbf{z})\, \mu^{(0)}(\mathrm{d}\mathbf{z}) \right| \\ &\quad\quad + \left| \int_{\mathbb{G}} f \circ \mathbf{T}^{m}(\mathbf{z})\, \mu^{(0)}(\mathrm{d}\mathbf{z}) - \int_{\mathbf{G}} f \circ \mathbf{T}^{\infty}(\mathbf{z})\, \mu^{(0)}(\mathrm{d}\mathbf{z}) \right| \\ &\quad = I + II + III. \end{aligned}$$

We deal with I, II and III separately.

Observe that

$$\begin{aligned} I &\leq \int_{\mathbb{G}} |f \circ \mathbf{T}^{\infty}(\mathbf{z}) - f \circ \mathbf{T}^{m}(\mathbf{z})|\, \mathbb{1}\{d_{\infty,\mathbb{Z}}(\mathbf{T}^{\infty}(\mathbf{z}), \mathbf{T}^{m}(\mathbf{z})) \leq \beta\}\, \mu_t^{(0)}(\mathrm{d}\mathbf{z}) \\ &\quad + \int_{\mathbb{G}} |f \circ \mathbf{T}^{\infty}(\mathbf{z}) - f \circ \mathbf{T}^{m}(\mathbf{z})|\, \mathbb{1}\{d_{\infty,\mathbb{Z}}(\mathbf{T}^{\infty}(\mathbf{z}), \mathbf{T}^{m}(\mathbf{z})) > \beta\}\, \mu_t^{(0)}(\mathrm{d}\mathbf{z}) \\ &\leq \omega_f(\beta)\mu_t^{(0)}(\mathbb{G}) + 2\mu_t^{(0)}(\{\mathbf{z} : d_{\infty,\mathbb{Z}}(\mathbf{T}^{\infty}(\mathbf{z}), \mathbf{T}^{m}(\mathbf{z})) > \beta\}). \end{aligned}$$

The first term above goes to 0 as $\beta \to \infty$ as $\mu_t^{(0)}(\mathbb{G})$ is finite for all large t while the second term goes to 0 by Lemma 3.7 by first letting $t \to \infty$ and then letting $m \to \infty$.

For any fixed m, $f \circ \mathbf{T}^m$ is continuous on $\mathbb{O}_0$ and hence on $\mathbb{G}$ and so by part (ii) of Theorem 2.1 and using Theorem 3.4 for $j = 0$, we have that II goes to 0 as $t \to \infty$ for every m.

To deal with III, first observe that for any $\mathbf{z} \in \mathbb{C}_{=1}$, $\lim_{m\to\infty} \mathbf{T}^m(\mathbf{z}) = \mathbf{T}^{\infty}(\mathbf{z})$. To see this let $\mathbf{z} = \lambda \mathbf{e}_i$. Then

$$d_{\infty,\mathbb{Z}}(\mathbf{T}^m(\mathbf{z}), \mathbf{T}^{\infty}(\mathbf{z})) \leq \lambda \sum_{l=m+1}^{\infty} \psi_l,$$

which goes to 0 as $m \to \infty$ as $\sum_{j=0}^{\infty} \psi_j < \infty$ by (C1). Since $\mu^{(0)}$ is finite on $\mathbb{G}$ and concentrates on $\mathbb{C}_{=1}$ and f is continuous and bounded, we have, by dominated convergence, that III goes to 0 as $m \to \infty$. □

Remark. The entire exercise in this paper could have been carried out in modestly more generality by assuming that the iid sequence $(Z_i, i \in \mathbb{Z})$ were real-valued and instead of (3.3) we assumed that $|Z_0|$ was regularly varying with tail index $\alpha > 0$ and

$$\lim_{t\to\infty} \frac{P[Z_0 > t]}{P[|Z_0| > t]} = p \text{ and } \lim_{t\to\infty} \frac{P[Z_0 < t]}{P[|Z_0| > t]} = 1 - p.$$

References

1. Patrick Billingsley, *Convergence of probability measures*, second ed., John Wiley & Sons Inc., New York, 1999, A Wiley-Interscience Publication. MR 2000e:60008
2. N.H. Bingham, C.M. Goldie, and J.L. Teugels, *Regular variation*, Cambridge University Press, 1987.
3. P.J. Brockwell and R.A. Davis, *Time series: Theory and methods*, second ed., Springer-Verlag, New York, 1991.
4. D.B.H. Cline, *Estimation and linear prediction for regression, autoregression and ARMA with infinite variance data*, Ph.D. thesis, Colorado State University, Fort Collins, CO 80521 USA, 1983.
5. ______, *Infinite series of random variables with regularly varying tails*, Tech. report, Inst. Appl. Math. Statist., Univ. British Columbia, 1983.
6. Daryl J. Daley and David Vere-Jones, *An introduction to the theory of point processes: volume I: elementary theory and methods*, vol. 1, Springer, 2007.
7. Bikramjit Das, Abhimanyu Mitra, and Sidney I. Resnick, *Living on the multidimensional edge: seeking hidden risks using regular variation*, Advances in Applied Probability **45** (2013), no. 1, 139–163.
8. Richard Davis and Sidney I. Resnick, *Limit theory for moving averages of random variables with regularly varying tail probabilities*, The Annals of Probability (1985), 179–195.
9. Henrik Hult and Filip Lindskog, *Extremal behavior of regularly varying stochastic processes*, Stochastic Processes and their applications **115** (2005), no. 2, 249–274.
10. ______, *Regular variation for measures on metric spaces*, Publ. Inst. Math.(Beograd)(NS) **80** (2006), no. 94, 121–140.
11. ______, *Extremal behavior of stochastic integrals driven by regularly varying lévy processes*, The Annals of Probability **35** (2007), no. 1, 309–339.
12. Henrik Hult and Gennady Samorodnitsky, *Tail probabilities for infinite series of regularly varying random vectors*, Bernoulli **14** (2008), no. 3, 838–864.
13. Olav Kallenberg, *Random measures*, Akademie-Verlag (Berlin and London and New York), 1983.
14. Filip Lindskog, Sidney I. Resnick, and Joyjit Roy, *Regularly varying measures on metric spaces: Hidden regular variation and hidden jumps*, Tech. report, School of ORIE, Cornell University, 2013.
15. Krishanu Maulik and Sidney I. Resnick, *Characterizations and examples of hidden regular variation*, Extremes **7** (2004), no. 1, 31–67.
16. Thomas Mikosch and Gennady Samorodnitsky, *The supremum of a negative drift random walk with dependent heavy-tailed steps*, The Annals of Applied Probability (2000), 1025–1064.
17. Sidney I. Resnick, *Extreme values, regular variation, and point processes, volume 4 of applied probability. a series of the applied probability trust*, Springer-Verlag, New York, 1987.
18. ______, *Hidden regular variation, second order regular variation and asymptotic independence*, Extremes **5** (2002), no. 4, 303–336.
19. ______, *Heavy-tail phenomena: probabilistic and statistical modeling*, Springer, 2007.
20. ______, *Extreme values, regular variation and point processes*, Springer, New York, 2008, Reprint of the 1987 original. MR MR2364939
21. Sidney I. Resnick and Eric Willekens, *Moving averages with random coefficients and random coefficient autoregressive models*, Stochastic Models **7** (1991), no. 4, 511–525.
22. Holger Rootzén, *Extremes of moving averages of stable processes*, Ann. Probab. **6** (1978), no. 5, 847–869. MR MR0494450 (58 #13311)
23. ______, *Extreme value theory for moving average processes*, Ann. Probab **14** (1986), no. 2, 612–652.
24. Dingcheng Wang and Qihe Tang, *Tail probabilities of randomly weighted sums of random variables with dominated variation*, Stochastic models **22** (2006), no. 2, 253–272.

Sidney I. Resnick, School of ORIE, Cornell University, Ithaca, NY 14853
E-mail address: `sir1@cornell.edu`

Joyjit Roy, School of ORIE, Cornell University, Ithaca, NY 14853
E-mail address: `jr653@cornell.edu`