Trimmed Lévy Processes and their Extremal Components

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Abstract

We analyse a trimmed stochastic process of the form ${}^{(r)}X_t = X_t - \sum_{i=1}^{r} \Delta_t^{(i)}$, where $(X_t)_{t\geq 0}$ is a driftless subordinator on $\mathbb{R}$ with its jumps on $[0,t]$ ordered as $\Delta_t^{(1)} \geq \Delta_t^{(2)} \cdots$. When $r \to \infty$, both ${}^{(r)}X_t \downarrow 0$ and $\Delta_t^{(r)} \downarrow 0$ a.s. for each $t > 0$, and it is interesting to study the weak limiting behaviour of $\big({}^{(r)}X_t, \Delta_t^{(r)}\big)$ in this case. We term this "large-trimming" behaviour. Concentrating on the case $t = 1$, we study joint convergence of $\big({}^{(r)}X_1, \Delta_1^{(r)}\big)$ under linear normalization, assuming extreme value-related conditions on the Lévy measure of X which guarantee that $\Delta^{(r)}$ has a limit distribution with linear normalization. Allowing ${}^{(r)}X$ to have random centering and scaling in a natural way, we show that $\big({}^{(r)}X_1, \Delta_1^{(r)}\big)$ has a bivariate normal limiting distribution, as $r \to \infty$; but replacing the random normalizations with natural deterministic ones produces non-normal limits which we can specify.

Keywords: Trimmed Lévy process, trimmed subordinator, subordinator large jumps, extreme value-related conditions, large-trimming limits.
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1 Introduction

Suppose $(X_t)_{t\geq 0}$ is a driftless subordinator with infinite Lévy measure Π and tail function $\overline{\Pi}(x) := \Pi(x,\infty)$, $x > 0$. Thus, (X_t) has Laplace transform $Ee^{-\lambda X_t} = e^{-t\psi(\lambda)}$, $t \geq 0$, where

$$\psi(\lambda) = \int_{(0,\infty)} (1 - e^{-\lambda x})\Pi(\mathrm{d}x),\ \lambda > 0.$$

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Let $\Delta_t^{(r)}$ be the rth largest jump of X_t on $[0,t]$, $t>0$, $r \in \mathbb{N} := \{1,2,\ldots\}$. The trimmed subordinator is defined to be ${}^{(r)}X_t = X_t - \sum_{i=1}^r \Delta_t^{(i)}$, $t>0$, $r \in \mathbb{N}$. In Buchmann et al. (2018); Ipsen et al. (2018) we considered distributional properties of $\Delta_t^{(r)}$ as a function of r and here we continue that study by considering the joint weak limiting behaviour of $\big({}^{(r)}X_t, \Delta_t^{(r)}\big)$ as $r \to \infty$. As $r \to \infty$, ${}^{(r)}X_t \downarrow 0$ and $\Delta_t^{(r)} \downarrow 0$ a.s. for each $t>0$, but conditionally on $\Delta_t^{(r)}$ we may consider ${}^{(r)}X_t$ as a Lévy process with Lévy measure restricted to $(0, \Delta_t^{(r)})$ (e.g., Resnick (1986)). So as $r \to \infty$ and big jumps are removed from ${}^{(r)}X$, it makes sense that we expect ${}^{(r)}X$ should have a Gaussian weak limit after centering and norming. We focus on the case $t=1$ and write simply $\big({}^{(r)}X, \Delta^{(r)}\big)$ for $\big({}^{(r)}X_1, \Delta_1^{(r)}\big)$ (the case of general $t>0$ is considered briefly in Section 7).

The approach we take is to assume conditions on $\overline{\Pi}$ guaranteeing that $\Delta^{(r)}$ has a limit distribution under linear normalization, and then prove that a normal limit distribution of ${}^{(r)}X$ conditional on the value of $\Delta^{(r)}$ also exists as $r \to \infty$. For finite r, we denote the conditional distribution with the notation ${}^{(r)}X|\Delta^{(r)}$. The conditioned limit of ${}^{(r)}X|\Delta^{(r)}$ initially requires a natural random centering and random scaling to achieve asymptotic normality. Having derived that, we then investigate replacing the random centering and scaling with deterministic versions.

According to (Buchmann et al., 2018, Section 4.2), there exist scaling functions $a_r > 0$ and centering functions $b_r \in \mathbb{R}$ such that, as $r \to \infty$, weak convergence[1] holds in $\mathbb{R}$:

$$\frac{\Delta^{(r)} - b_r}{a_r} \Rightarrow \Delta^{(\infty)}, \tag{1.1}$$

with $\Delta^{(\infty)}$ non-degenerate, if, for $x \in \mathbb{R}$ such that $a_r x + b_r > 0$,

$$\lim_{r\to\infty} \frac{r - \overline{\Pi}\big(a_r x + b_r\big)}{\sqrt{r}} = h(x), \tag{1.2}$$

where $h(x) \in \mathbb{R}$ is a non-decreasing and non-constant limit function. The function $h(x)$ has the form (see Buchmann et al. (2018), Eq. (4.2)):

$$\tfrac{1}{2}h(x) = \tfrac{1}{2}h_\gamma(x) = \begin{cases} -\frac{1}{\gamma}\log(1-\gamma x), & \text{if } \gamma \in \mathbb{R}\setminus\{0\},\ 1-\gamma x > 0, \\ x, & \text{if } \gamma = 0,\ x \in \mathbb{R}. \end{cases} \tag{1.3}$$

We can identify the distribution of the limit random variable $\Delta^{(\infty)}$ in terms of the inverse function $h^{\leftarrow}$ of h. From (1.3) this function satisfies, for $y \in \mathbb{R}$,

$$h^{\leftarrow}(y) = \frac{1 - e^{-\gamma y/2}}{\gamma} = \begin{cases} y/2, & \text{if } \gamma = 0, \\ \dfrac{1 - e^{-\gamma y/2}}{\gamma}, & \text{if } \gamma > 0, \\ \dfrac{e^{|\gamma| y/2} - 1}{|\gamma|}, & \text{if } \gamma < 0. \end{cases} \tag{1.4}$$

[1]We use the symbol "$\Rightarrow$" to denote weak convergence in $\mathbb{R}$ or $\mathbb{R}^2$.

We note that $h^{\leftarrow} : \mathbb{R} \mapsto \mathbb{R}_\gamma$, where, for $\gamma \in \mathbb{R}$,

$$\mathbb{R}_\gamma := \{x \in \mathbb{R} : 1 - \gamma x > 0\} = \begin{cases} \mathbb{R}, & \text{if } \gamma = 0, \\ \big(-\infty, \frac{1}{\gamma}\big), & \text{if } \gamma > 0, \\ \big(-\frac{1}{|\gamma|}), \infty\big), & \text{if } \gamma < 0. \end{cases}$$

Taking inverses in (1.2), we get an equivalent form

$$\lim_{r\to\infty} \frac{\overline{\Pi}^{\leftarrow}(r - y\sqrt{r}) - b_r}{a_r} = h^{\leftarrow}(y), \qquad y \in \mathbb{R}, \tag{1.5}$$

where the inverse function $\overline{\Pi}^{\leftarrow}$ to $\overline{\Pi}$ is defined by

$$\overline{\Pi}^{\leftarrow}(x) = \{\inf y > 0 : \overline{\Pi}(y) \le x\}.$$

From (1.4) we have $h^{\leftarrow}(0) = 0$, so from (1.5) we deduce for $y \in \mathbb{R}$,

$$\begin{aligned} \lim_{r\to\infty} \frac{\overline{\Pi}^{\leftarrow}(r - y\sqrt{r}) - \overline{\Pi}^{\leftarrow}(r)}{a_r} &= \lim_{r\to\infty} \Big(\frac{\overline{\Pi}^{\leftarrow}(r - y\sqrt{r}) - b_r}{a_r} - \frac{\overline{\Pi}^{\leftarrow}(r) - b_r}{a_r} \Big) \\ &= h^{\leftarrow}(y) - h^{\leftarrow}(0) = h^{\leftarrow}(y). \end{aligned} \tag{1.6}$$

We conclude that for centering constants we may always set $b_r = \overline{\Pi}^{\leftarrow}(r)$ (appropriate norming constants a_r will be specified later).

The convergences in (1.2), (1.5) and (1.6) are locally uniform since they are convergences of monotone functions to a continuous limit. Recalling the notation in (1.1), we have under (1.2) that

$$\lim_{r\to\infty} \mathrm{P}\Big(\frac{\Delta^{(r)} - b_r}{a_r} \le x\Big) = \mathrm{P}\big(\Delta^{(\infty)} \le x\big) = \Phi\big(h(x)\big), \; x \in \mathbb{R}, \tag{1.7}$$

where $\Phi(x)$ is the standard normal cdf. (This will be proved in (3.1) below, or see Buchmann et al. (2018)). Thus $\Delta^{(\infty)} \stackrel{\mathrm{D}}{=} h^{\leftarrow}(N(0,1))$ where $N(0,1)$ is a standard normal random variable.

Remark 1.1. Since we assume only positive jumps for the Lévy process, $\Pi(\cdot)$ concentrates on $(0,\infty)$. This implies that the case $\gamma > 0$ in (1.3) or (1.4) cannot occur. From the discussion in Buchmann et al. (2018), (1.2) means that the function $G(x) := e^{-\sqrt{\overline{\Pi}(x)}}$ defined on $(0,\infty)$ is a distribution function in the minimal domain of attraction, which for the $\gamma > 0$ case would require $G(x)$ to be regularly varying as $x \to -\infty$. This is impossible because $\Pi(\cdot)$ concentrates on $\mathbb{R}_+$. So from now on we concentrate attention on the cases $\gamma \le 0$.

To conclude this introduction we set out the steps we intend to follow to understand the joint limit behaviour of $({}^{(r)}X, \Delta^{(r)})$ as $r \to \infty$ under (1.2) or, equivalently, (1.5).

1. As discussed, we expect a normal limit as $r \to \infty$ for ${}^{(r)}X$ with suitable linear normalizations. We show that this happens for ${}^{(r)}X|\Delta^{(r)}$ under a natural *random* centering and scaling (Theorem 2.1).

2. Following that, we extend asymptotic normality of ${}^{(r)}X|\Delta^{(r)}$ to a joint asymptotic weak limit for $({}^{(r)}X, \Delta^{(r)})$ in which the limit has independent components. At this stage, ${}^{(r)}X$ still has random centering and scaling, though $\Delta^{(r)}$ has non-random normalizations (Corollary 2.1).

3. Finally, we note there is a cost to replacing the random centering and scaling: dependencies and non-normality are introduced into the limit (Theorems 2.2 and 2.3).

In the next section we give our main results. Proofs of the theorems and further discussion are deferred to Sections 5 and 6. A number of subsidiary propositions are also needed; these are proved in Sections 3 and 4. Section 7 concludes with some general discussion.

2 Main Results

Throughout, we write $\mathrm{P}^{\Delta^{(r)}}(\cdot) = \mathrm{P}(\cdot|\Delta^{(r)})$ for the conditional distribution, given $\Delta^{(r)}$. In introducing this we make the simplifying assumption that Π is atomless (equivalently, $\overline{\Pi}$ is continuous on $(0,\infty)$). This means that the inverse function $\overline{\Pi}^{\leftarrow}$ is strictly increasing on $(0,\infty)$ and the ordered jumps $\Delta X_t^{(i)}$ are uniquely defined. We expect that this assumption can be removed by some well known manipulations which would add little of interest to the exposition, so we omit them.

We will also need truncated first and second moment functions, defined for $t > 0$ by

$$\mu(t) = \int_0^t x\,\Pi(\mathrm{d}x) \quad \text{and} \quad \sigma^2(t) = \int_0^t x^2\,\Pi(\mathrm{d}x). \tag{2.1}$$

Theorem 2.1. *Suppose X is a driftless subordinator on $(0,\infty)$ with Lévy measure $\Pi(\cdot)$ on $(0,\infty)$ that satisfies* (1.2) *or, equivalently,* (1.5)*, for deterministic functions $a_r > 0$ and $b_r \in \mathbb{R}$. Then we have*

$$\lim_{r\to\infty} \mathrm{P}^{\Delta^{(r)}}\left(\frac{{}^{(r)}X - \mu(\Delta^{(r)})}{\sigma(\Delta^{(r)})} \le x\right) = \Phi(x),\ x \in \mathbb{R}. \tag{2.2}$$

Remark 2.1. By the dominated convergence theorem the convergence in (2.2) holds unconditionally as well, so we also have

$$\frac{{}^{(r)}X - \mu(\Delta^{(r)})}{\sigma(\Delta^{(r)})} \Rightarrow N(0,1),\ \text{as } r \to \infty,$$

under the conditions of Theorem 2.1.

Retaining the random centering and scaling, Theorem 2.1 immediately leads to a joint limit distribution for $({}^{(r)}X, \Delta^{(r)})$. In the following corollary, N_X and N_Γ are independent standard normal random variables, being the limits of the standardised ${}^{(r)}X$ and $\Delta^{(r)}$, with the subscripts on N_X and N_Γ serving to distinguish the components corresponding to ${}^{(r)}X$ and $\Delta^{(r)}$. (Throughout, N_X and N_Γ will be independent standard normal random variables corresponding to ${}^{(r)}X$ and $\Delta^{(r)}$ in this way.)

Corollary 2.1. *Under the conditions leading to* (1.7) *and* (2.2) *we have, in* $\mathbb{R}^2$,

$$\Big(\frac{{}^{(r)}X - \mu(\Delta^{(r)})}{\sigma(\Delta^{(r)})}, \frac{\Delta^{(r)} - b_r}{a_r}\Big) \Rightarrow \big(N_X, h^{\leftarrow}(N_\Gamma)\big), \text{ as } r \to \infty.$$

Next we need to understand the effect of replacing the random centering and scaling by deterministic counterparts. We begin with the scaling constants. The treatment is broken up according to the cases of the constant γ in (1.3).

Theorem 2.2. *Suppose* (1.2) *holds.*

(i) *When* $\gamma < 0$, *we have, as* $r \to \infty$, *with* $b_r = \overline{\Pi}^{\leftarrow}(r)$,

$$\Big(\frac{{}^{(r)}X - \mu(\Delta^{(r)})}{\sigma(b_r)}, \frac{\Delta^{(r)}}{b_r}\Big) \Rightarrow \Big(N_X e^{-N_\Gamma|\gamma|/2}, e^{-N_\Gamma|\gamma|/2}\Big) \tag{2.3}$$

and removing the random centering from ${}^{(r)}X$ *gives*

$$\Big(\frac{{}^{(r)}X - \mu(b_r)}{b_r\sqrt{r}}, \frac{\Delta^{(r)}}{b_r}\Big) \Rightarrow \Big(\frac{2}{|\gamma|}(e^{-N_\Gamma|\gamma|/2} - 1), e^{-N_\Gamma|\gamma|/2}\Big), \text{ as } r \to \infty. \tag{2.4}$$

(ii) *When* $\gamma = 0$, *we have, as* $r \to \infty$, *with* $a_r = 2(\overline{\Pi}^{\leftarrow}(r - \sqrt{r}) - \overline{\Pi}^{\leftarrow}(r))$ *and* $b_r = \overline{\Pi}^{\leftarrow}(r)$,

$$\Big(\frac{{}^{(r)}X - \mu(\Delta^{(r)})}{\sigma(b_r)}, \frac{\Delta^{(r)} - b_r}{a_r}\Big) \Rightarrow \Big(N_X, \frac{N_\Gamma}{2}\Big), \text{ as } r \to \infty. \tag{2.5}$$

Remark 2.2. (a) Note that when $\gamma < 0$, we no longer have independence of the components in the limit when we replace the random scaling by the deterministic one as in (2.3) and (2.4).

(b) When $\gamma = 0$, we can always make the scaling deterministic, as in (2.5), however this is not in general the case for the centering; replacing $\mu(\Delta^{(r)})$ with $\mu(b_r)$ in (2.5) is only possible under some subsidiary conditions. A detailed discussion is given in Section 6. For the special case when $\overline{\Pi} \in RV_0(-\alpha)$ for $0 \le \alpha \le 1$, the joint limiting distribution of ${}^{(r)}X$ and $\Delta^{(r)}$ is specified in the following theorem.

Theorem 2.3. *Suppose* $\gamma = 0$ *and* $\overline{\Pi}$ *is regularly varying at* 0 *with index* $-\alpha$. *Let* $c_\alpha := \alpha/(2-\alpha)$, $a_r = 2(\overline{\Pi}^{\leftarrow}(r - \sqrt{r}) - \overline{\Pi}^{\leftarrow}(r))$ *and* $b_r = \overline{\Pi}^{\leftarrow}(r)$.

(i) Suppose $0 < c_\alpha \le 1$*, so that* $\alpha \le 1$*. Then*

$$\Big(\frac{{}^{(r)}X - \mu(b_r)}{\sigma(b_r)}, \frac{\Delta^{(r)} - b_r}{a_r}\Big) \Rightarrow \Big(N_X + \frac{N_\Gamma}{\sqrt{c_\alpha}}, \frac{N_\Gamma}{2}\Big). \tag{2.6}$$

(ii) Suppose $c_\alpha = 0$*, so that* $\overline{\Pi}$ *is slowly varying at 0. Then*

$$\Big(\frac{{}^{(r)}X - \mu(b_r)}{b_r\sqrt{r}}, \frac{\Delta^{(r)} - b_r}{a_r}\Big) \Rightarrow \Big(N_X, \frac{N_\Gamma}{2}\Big).$$

3 Convergence of $\Delta^{(r)}$

We begin the program outlined in the previous section by examining the convergence of $\Delta^{(r)}$, after norming and centering, as $r \to \infty$. Throughout this section assume[2] (1.2) or, equivalently, (1.5), and recall the function h in (1.3) and its inverse $h^{\leftarrow}$ in (1.4).

We need some more preliminary setting up. Let $\{\Gamma_l\}$ and $\{\Gamma'_l\}$ be cumulative sums of independent sequences of iid standard exponential random variables. We can construct the subordinator X from a Poisson random measure

$$\mathbb{X}(\cdot) = \sum_{l=1}^{\infty} \delta_{\overline{\Pi}^{\leftarrow}(\Gamma_l)}$$

where the mean measure is $\Pi(\cdot)$ and the points are written in decreasing order Ferguson & Klass (1972), LePage et al. (1981), Resnick (1986), Buchmann et al. (2016) and (Resnick, 2008, p.139, Ex. 3.38). This means

$$X = \int_0^\infty x\,\mathbb{X}(\mathrm{d}x) = \sum_{l=1}^{\infty} \overline{\Pi}^{\leftarrow}(\Gamma_l) \quad \text{and} \quad \Delta^{(r)} = \overline{\Pi}^{\leftarrow}(\Gamma_r);$$

also

$${}^{(r)}X = \sum_{l=r+1}^{\infty} \overline{\Pi}^{\leftarrow}(\Gamma_l) = \sum_{l=1}^{\infty} \overline{\Pi}^{\leftarrow}(\Gamma_r + \Gamma'_l).$$

3.1 Proof of the convergence in (1.7)

We may understand the form of the limit for $\Delta^{(r)}$ in (1.7) as follows. By properties of the gamma distribution, we know that, as $r \to \infty$, $G_r := (\Gamma_r - r)/\sqrt{r} \Rightarrow N(0,1)$, a standard normal random variable. Assume (1.2), or, equivalently, (1.5); then, owing to the local uniform convergence in (1.5), we get (1.7) from

$$\frac{\Delta^{(r)} - b_r}{a_r} = \frac{\overline{\Pi}^{\leftarrow}(\Gamma_r) - b_r}{a_r} = \frac{\overline{\Pi}^{\leftarrow}(r + G_r\sqrt{r}) - b_r}{a_r} \Rightarrow h^{\leftarrow}(-N(0,1)) \stackrel{\mathrm{D}}{=} h^{\leftarrow}(N(0,1)). \tag{3.1}$$

[2] The simplifying assumption that $\overline{\Pi}$ is continuous on $(0,\infty)$ is not in fact needed for the results in this section.

3.2 Role of the de Haan classes $\mathbf{\Gamma}$ and $\mathbf{\Pi}$

Now introduce the function $H : [0,\infty) \mapsto [1,\infty)$ defined by

$$H(t) = e^{2\sqrt{t}}, t > 0, \tag{3.2}$$

and define the non-increasing function V by

$$V(x) = \overline{\Pi}^{\leftarrow} \circ H^{\leftarrow}(x), \quad x > 1, \tag{3.3}$$

and changing variables gives the representation $\overline{\Pi}^{\leftarrow}(x) = V(H(x))$,

The function H is the canonical example of a non-decreasing function in the de Haan class $\mathbf{\Gamma}$ with auxiliary function $f(t) = \sqrt{t}$ Bingham et al. (1989); de Haan (1970, 1974); de Haan & Resnick (1973); Geluk & de Haan (1987); Resnick (2008) satisfying

$$\lim_{t\to\infty} \frac{H(t + xf(t))}{H(t)} = e^x, \quad x \in \mathbb{R}. \tag{3.4}$$

This can be verified directly or by reference to (de Haan, 1974, p. 248, line -1). The inverse function $H^{\leftarrow} : [1,\infty) \mapsto [0,\infty)$ to H is $H^{\leftarrow}(y) = \frac{1}{4}\log^2 y$, $y > 1$, and inverting (3.4) shows that $H^{\leftarrow}$ satisfies

$$\lim_{s\to\infty} \frac{H^{\leftarrow}(sy) - H^{\leftarrow}(s)}{f(H^{\leftarrow}(s))} = \log y, \quad y > 0, \tag{3.5}$$

so $H^{\leftarrow}$ is an increasing function in de Haan's function class $\mathbf{\Pi}$ (Bingham et al. (1989); de Haan (1970); Resnick (2008) or (de Haan & Ferreira, 2006, p. 375)). It has slowly varying auxiliary function $g(s) = f \circ H^{\leftarrow}(s) = \sqrt{H^{\leftarrow}(s)} = \frac{1}{2}\log s$ which is the denominator in (3.5). The convergence in (3.5) is uniform in compact intervals of y bounded away from 0.

Recall that (1.5) is in force throughout, so we have (1.6) also. Applying the uniform convergence in (1.6) and (3.5), we see that V satisfies, for $x > 0$,

$$\begin{aligned}
\lim_{s\to\infty} \frac{V(sx) - V(s)}{a \circ H^{\leftarrow}(s)} &= \lim_{s\to\infty} \frac{\overline{\Pi}^{\leftarrow} \circ H^{\leftarrow}(sx) - \overline{\Pi}^{\leftarrow} \circ H^{\leftarrow}(s)}{a \circ H^{\leftarrow}(s)} \\
&= \lim_{s\to\infty} \frac{\overline{\Pi}^{\leftarrow}\left(H^{\leftarrow}(s) + \left\{\frac{H^{\leftarrow}(sx) - H^{\leftarrow}(s)}{\sqrt{H^{\leftarrow}(s)}}\right\}\sqrt{H^{\leftarrow}(s)}\right) - \overline{\Pi}^{\leftarrow} \circ H^{\leftarrow}(s)}{a \circ H^{\leftarrow}(s)} \\
&= \lim_{t\to\infty} \frac{\overline{\Pi}^{\leftarrow}(t + \log x \cdot \sqrt{t}) - \overline{\Pi}^{\leftarrow}(t)}{a(t)} \\
&= h^{\leftarrow}(-\log x).
\end{aligned}$$

Thus, for $x > 0$, using the form of $h^{\leftarrow}$ in (1.4),

$$\lim_{s\to\infty} \frac{V(sx) - V(s)}{a \circ H^{\leftarrow}(s)} = \begin{cases} -\frac{1}{2}\log x, & \text{if } \gamma = 0, \\ -\frac{1}{2}\left(\frac{x^{\gamma/2} - 1}{\gamma/2}\right), & \text{if } \gamma \neq 0. \end{cases} \tag{3.6}$$

From (de Haan & Ferreira, 2006, Theorem B.2.1, p.372), we get $a \circ H^{\leftarrow}(s) \in RV_{\gamma/2}$. Then multiply the limit relation in (3.6) by -1 to see that the non-decreasing function $-V$ is extended regularly varying at ∞ ((de Haan & Ferreira, 2006, p.127ff, p.139)).

We summarise the working up to (3.6) as follows.

Proposition 3.1. *Assume* (1.5).

(i) *When* $\gamma < 0$*:*

$$V(x) = -V(\infty) - (-V(x)) \sim a \circ H^{\leftarrow}(x)/|\gamma| \in RV_{-|\gamma|/2}, \text{ as } x \to \infty; \quad (3.7)$$

(ii) *When* $\gamma = 0$*:* $-V \in \mathbf{\Pi}$ *(or, equivalently,* $V \in \mathbf{\Pi}_{-}$*, de Haan & Resnick (1979)) with slowly varying auxiliary function* $\frac{1}{2} a \circ H^{\leftarrow}$*.*

Remark 3.1. (i) Note V being regularly varying with negative index in (3.7) is consistent with V being non-increasing.

(ii) Note that $\gamma > 0$ cannot obtain in (3.6). The numerator on the left side of the limit is a difference of two decreasing functions which as functions of s approach 0. The denominator is regularly varying with *positive* index and hence asymptotically increasing. So we cannot get a non-trivial limit. See Remark 1.1.

3.3 Refining the centering and scaling for $\Delta^{(r)}$.

Now we apply the material from Subsection 3.2 to refine the centering and scaling for $\Delta^{(r)}$. Recall that (1.2) or equivalently (1.5) is in force throughout. Depending on the range of γ, we may now simplify the form of the limit law for $\Delta^{(r)}$ as follows.

Proposition 3.2. *Suppose* (1.2) *holds with* $h^{\leftarrow}(y)$ *as in* (1.4)*. Let* N_Γ *be a standard normal random variable.*

(i) *When* $\gamma < 0$*: we may take* $b_r = \overline{\Pi}^{\leftarrow}(r)$*, and then*

$$\frac{\Delta^{(r)}}{b_r} \Rightarrow e^{-N_\Gamma |\gamma|/2}, \text{ as } r \to \infty. \quad (3.8)$$

(ii) *When* $\gamma = 0$*: we may take* $b_r = \overline{\Pi}^{\leftarrow}(r)$ *and* $a_r = 2(\overline{\Pi}^{\leftarrow}(r - \sqrt{r}) - \overline{\Pi}^{\leftarrow}(r))$*, and then*

$$\frac{\Delta^{(r)} - b_r}{a_r} \Rightarrow \frac{N_\Gamma}{2}, \text{ as } r \to \infty. \quad (3.9)$$

Furthermore

$$a_r = o(b_r) \text{ and } \frac{\Delta^{(r)}}{b_r} \Rightarrow 1, \text{ as } r \to \infty. \quad (3.10)$$

Proof of Proposition 3.2: (i) Take $\gamma < 0$. From (3.7), $V(x) \sim a \circ H^{\leftarrow}(x)/|\gamma|$, so

$$b_r = \overline{\Pi}^{\leftarrow}(r) = V(H(r)) \sim a_r/|\gamma|. \tag{3.11}$$

Thus (3.1) can be written

$$\frac{\Delta^{(r)} - b_r}{|\gamma| b_r} \Rightarrow h^{\leftarrow}(N_\Gamma),$$

and hence, using (1.4),

$$\frac{\Delta^{(r)}}{b_r} \Rightarrow 1 + |\gamma| h^{\leftarrow}(N_\Gamma) \stackrel{\mathrm{D}}{=} e^{-|\gamma| N_\Gamma/2},$$

which gives (3.8).

(ii) Take $\gamma = 0$. From (1.4) with $\gamma = 0$ and (1.5) with $y = 1$ we get

$$\frac{2(\overline{\Pi}^{\leftarrow}(r - \sqrt{r}) - \overline{\Pi}^{\leftarrow}(r))}{a_r} \to 1,$$

and the choice of a_r for (3.9) follows from the convergence to types theorem. Since $V \in \mathbf{\Pi}_-$ with auxiliary function $a \circ H^{\leftarrow}$ and the ratio of a non-negative $\mathbf{\Pi}$ function to its auxiliary function tends to ∞, we have

$$\lim_{r\to\infty} \frac{b_r}{a_r} = \lim_{r\to\infty} \frac{\overline{\Pi}^{\leftarrow}(r)}{a_r} = \lim_{r\to\infty} \frac{V(H((r))}{a \circ H^{\leftarrow} \circ H(r)} = \infty.$$

Finally, dividing (3.9) by b_r/a_r, which tends to ∞ as $r \to \infty$, yields a limit of 0 which is tantamount to saying $\Delta^{(r)}/b_r \Rightarrow 1$. $\square$

Example 1. *[Stable Subordinator]*

To fix ideas, consider the case of the stable subordinator, where

$$\overline{\Pi}(x) = x^{-\alpha},\ x > 0,\ 0 < \alpha < 1, \qquad \overline{\Pi}^{\leftarrow}(y) = y^{-1/\alpha},\ y > 0. \tag{3.12}$$

The numerator of the left side of (1.5) is then, for $y \in \mathbb{R}$,

$$\begin{aligned}(r - y\sqrt{r})^{-1/\alpha} - r^{-1/\alpha} &= r^{-1/\alpha}\Big(\Big(1 - \frac{y}{\sqrt{r}}\Big)^{-1/\alpha} - 1\Big) \\ &\sim r^{-1/\alpha - 1/2} y/\alpha,\ \text{as } r \to \infty,\end{aligned}$$

so (1.5) holds if we take

$$b_r = r^{-1/\alpha}, \quad a_r = 2r^{-1/\alpha-1/2}/\alpha, \quad \text{and} \quad h^{\leftarrow}(y) = y/2.$$

Thus we are in the $\gamma = 0$ case.

Furthermore, recalling that $H^{\leftarrow}(x) = \frac{1}{4}\log^2 x$, we get

$$V(x) = \overline{\Pi}^{\leftarrow} \circ H^{\leftarrow}(x) = \Big(\frac{1}{4}\log^2 x\Big)^{-1/\alpha} \in \mathbf{\Pi}_-.$$

The auxiliary function corresponding to $\mathbf{\Pi}_-$-varying V is

$$a \circ H^{\leftarrow}(x) = \frac{2}{\alpha}\Big(\frac{1}{4}\log^2 x\Big)^{-1/\alpha-1/2},$$

which is slowly varying at ∞ (as it should be), and for $x > 0$

$$\lim_{s\to\infty} \frac{V(sx) - V(s)}{a \circ H^{\leftarrow}(s)} = -\frac{1}{2}\log x.$$

This completes the line-up of results needed for our analysis of $\Delta^{(r)}$. Next we turn to the results needed for ${}^{(r)}X$.

4 Further implications of the variation of $\overline{\Pi}^{\leftarrow}$

Here we derive additional properties of $\overline{\Pi}^{\leftarrow}$ depending on whether $\gamma < 0$ or $\gamma = 0$. These properties will be needed to replace random centerings and scalings for ${}^{(r)}X$ by deterministic normalizations in the following sections.

4.1 Case $\gamma < 0$.

Suppose throughout that (1.2) holds with $h(x) = h_\gamma(x)$ for $\gamma < 0$ as in (1.3), so by (3.8) and (3.11) we can take $a_r = |\gamma| b_r$ and $b_r = \overline{\Pi}^{\leftarrow}(r)$, and have $\Delta^{(r)}/b_r \Rightarrow e^{-N_\Gamma|\gamma|/2}$. Recall the distribution function G defined as $G(x) = e^{-\sqrt{\overline{\Pi}(x)}}$, $x > 0$. Then the following hold.

Proposition 4.1. *Assume* (1.2) *holds with* $\gamma < 0$.

(i) *For* $p \geq 1$,

$$\int_0^{b_r} u^p \Pi(\mathrm{d}u) \sim \frac{2}{p|\gamma|} b_r^p \sqrt{r}, \ \ as\ r \to \infty. \tag{4.1}$$

In particular, when $p = 2$,

$$\sigma^2(b_r) \sim \frac{1}{|\gamma|} b_r^2 \sqrt{r}, \ \ as\ r \to \infty.$$

(ii) $\overline{\Pi}(x)$ *is slowly varying at* 0, $\sigma^2(x)$ *is regularly varying at* 0 *with index 2 and* $G(x)$ *is regularly varying at 0 with index* $1/|\gamma|$.

Proof of Proposition 4.1: (i) Assume (1.2) holds and keep $\gamma < 0$ throughout. To see (i), use $\overline{\Pi}^{\leftarrow} = V \circ H$ from (3.3), where V is regularly varying at ∞ with index $\gamma/2$ and H is a $\mathbf{\Gamma}$ function with auxiliary function $f(t) = \sqrt{t}$. Such a composition is again in the class $\mathbf{\Gamma}$ (Bingham et al. (1989); de Haan (1970, 1974); de Haan & Ferreira (2006); Resnick (2007, 2008)), so for $z \in \mathbb{R}$

$$\frac{\overline{\Pi}^{\leftarrow}(r + \sqrt{r}z)}{\overline{\Pi}^{\leftarrow}(r)} = \frac{V\left(\frac{H(r+\sqrt{r}z)}{H(r)} H(r)\right)}{V(H(r))} \to e^{z\gamma/2},$$

or, equivalently, after a change of variable $w = -z|\gamma|/2$,

$$\frac{\overline{\Pi}^{\leftarrow}(r + \frac{2\sqrt{r}}{|\gamma|} w)}{\overline{\Pi}^{\leftarrow}(r)} \to e^{-w}, \quad w \in \mathbb{R}. \tag{4.2}$$

The limit relation (4.2) identifies the auxiliary function of the non-increasing $\mathbf{\Gamma}$-varying function $\overline{\Pi}^{\leftarrow}(x)$ as $f_1(r) = \frac{2}{|\gamma|}\sqrt{r}$. The function $\mathbf{\Gamma}$ class already appeared in (3.4) where we constructed a non-decreasing function in $\mathbf{\Gamma}$. Likewise for any $p \geq 1$, $(\overline{\Pi}^{\leftarrow})^p \in \mathbf{\Gamma}$ with auxiliary function $f_p(r) = \frac{2}{p|\gamma|}\sqrt{r}$. Auxiliary functions of $\mathbf{\Gamma}$-functions are unique up to asymptotic equivalence and also may be constructed in a canonical way (see for example, (de Haan & Ferreira, 2006, page 19, eqn. 1.2.5), (Bingham et al., 1989, p. 177, Corollary 3.10.5(b))). Therefore, we may identify the auxiliary function of the $\mathbf{\Gamma}$-function $(\overline{\Pi}^{\leftarrow})^p$ in two asymptotically equivalent ways:

$$f_p(r) \sim \frac{2}{p|\gamma|}\sqrt{r} \quad \text{or} \quad f_p(r) \sim \frac{\int_r^\infty \left(\overline{\Pi}^{\leftarrow}(u)\right)^p \mathrm{d}u}{\left(\overline{\Pi}^{\leftarrow}(r)\right)^p}, \quad (r \to \infty). \tag{4.3}$$

Using the transformation theorem for integrals, we can write (e.g. (Brémaud, 1981, p. 301))

$$\int_0^{\overline{\Pi}^{\leftarrow}(r)} u^p \Pi(\mathrm{d}u) = \int_r^\infty \left(\overline{\Pi}^{\leftarrow}(u)\right)^p \mathrm{d}u$$

and since $b_r = \overline{\Pi}^{\leftarrow}(r)$, applying (4.3) gives (4.1).

(ii) Invert the limit relation (4.2) and change variables $s = b_r \to 0$ to get

$$\lim_{s\to 0} \frac{\overline{\Pi}(sy) - \overline{\Pi}(s)}{(2/|\gamma|)\sqrt{\overline{\Pi}(s)}} = -\log y, \ \ y > 0.$$

Dividing by $\overline{\Pi}(t)$ instead of $\sqrt{\overline{\Pi}(t)}$, we get zero on the right side in the limit, which shows that $\overline{\Pi}(t)$ is slowly varying at 0. Factoring as

$$\overline{\Pi}(sy) - \overline{\Pi}(s) = (\overline{\Pi}^{1/2}(sy) - \overline{\Pi}^{1/2}(s))(\overline{\Pi}^{1/2}(sy) + \overline{\Pi}^{1/2}(s))$$

and using the slow variation of $\overline{\Pi}(x)$, hence of $\overline{\Pi}^{1/2}(x)$, at 0, gives the regular variation of $e^{-\sqrt{\overline{\Pi}(x)}}$ at 0 with index $1/|\gamma|$. □

4.2 Case $\gamma = 0$.

Suppose (1.2) holds with $h(x) = h_\gamma(x) = 2x$ for $\gamma = 0$ as in (1.3). From Proposition 3.2 we know in this case we may take $b_r = \overline{\Pi}^{\leftarrow}(r)$ and $a_r = 2(\overline{\Pi}^{\leftarrow}(r - \sqrt{r}) - b_r)$, and then $(\Delta^{(r)} - b_r)/a_r \Rightarrow N_\Gamma/2$, where N_Γ is a standard normal random variable. Also $a_r/b_r \to 0$ and $\Delta^{(r)}/b_r \Rightarrow 1$. The following proposition parallels Proposition 4.1 for the $\gamma = 0$ case. Recall the functions H from (3.2) and $V = \overline{\Pi}^{\leftarrow} \circ H^{\leftarrow}$ from (3.3), satisfying $V^{\leftarrow} = H \circ \overline{\Pi}$ and $V \in \mathbf{\Pi}_-$ with slowly varying auxiliary function $\frac{1}{2}a \circ H^{\leftarrow}(s)$.

Proposition 4.2. *Assume that* (1.2) *holds with* $\gamma = 0$.

(i) *For* $p \geq 1$, *there exist* $\mathbf{\Pi}$*-varying functions* $\pi_p(\cdot)$ *such that*

$$\int_0^{b_r} u^p \Pi(du) = \pi_p(H(r)) = \pi_p(e^{2\sqrt{r}}) \tag{4.4}$$

where the slowly varying auxiliary function of π_p *is* $g_p(t) = \frac{1}{2}V^p(t)\log t$.

(ii) *As* $r \to \infty$,

$$\frac{\sigma^2(\Delta^{(r)})}{\sigma^2(b_r)} \Rightarrow 1. \tag{4.5}$$

Proof of Proposition 4.2: (i) For $p \geq 1$ and $t > 0$, recall $H^{\leftarrow}(y) = \frac{1}{4}\log^2 y$ and consider

$$\begin{aligned}\int_0^t u^p \Pi(\mathrm{d}u) &= \int_{\overline{\Pi}(t)}^{\infty} \big(\overline{\Pi}^{\leftarrow}(s)\big)^p \mathrm{d}s = \int_{\overline{\Pi}(t)}^{\infty} \big(V \circ H(s)\big)^p ds = \int_{H\circ\overline{\Pi}(t)}^{\infty} V^p(v) dH^{\leftarrow}(v) \\ &= \int_{V^{\leftarrow}(t)}^{\infty} V^p(v)\frac{1}{2}\log v \frac{\mathrm{d}v}{v} = \pi_p\big(V^{\leftarrow}(t)\big),\end{aligned} \tag{4.6}$$

where we define

$$\pi_p(t) = \int_t^{\infty} V^p(v)\frac{1}{2}\log v \frac{\mathrm{d}v}{v}. \tag{4.7}$$

Now, V is $\mathbf{\Pi}$-varying and hence slowly varying, so V^p is slowly varying, as is $\log v$. Thus the function $\pi_p(\cdot)$ is the integral of a -1-varying function. The indefinite integral of a -1-varying function is $\mathbf{\Pi}$-varying (de Haan (1976); de Haan & Ferreira (2006), (Resnick, 2008, p. 30)). Thus $\pi_p \in \mathbf{\Pi}$ and the auxiliary function is $g_p(t) = \frac{1}{2}V^p(t)\log t$.

(ii) A $\mathbf{\Pi}$-varying function is always of larger order than its auxiliary function (de Haan & Ferreira, 2006, p.378), so

$$\lim_{t\to\infty} \frac{\pi_p(t)}{g_p(t)} = \infty. \tag{4.8}$$

Now we apply these results with $p = 2$. Because of the representation in (4.6), we invert the $\mathbf{\Pi}_{-}$-variation of $V(\cdot)$ in (3.6) and get for $y > 0$,

$$\frac{V^{\leftarrow}\big(b_r - ya_r\big)}{V^{\leftarrow}(b_r)} \to e^{2y}, \text{ as } r \to \infty. \tag{4.9}$$

To show (4.5), take the difference between numerator and denominator and use (4.6):

$$\sigma^2(\Delta^{(r)}) - \sigma^2(b_r) = \pi_2\big(V^{\leftarrow}(\Delta^{(r)})\big) - \pi_2\big(V^{\leftarrow}(b_r)\big).$$

From (3.1) write $(\Delta^{(r)} - b_r)/a_r = \xi_r$ so that $\xi_r \Rightarrow N_\Gamma/2$ and remember $b_r = \overline{\Pi}^{\leftarrow}(r)$. The previous difference then becomes

$$\pi_2\big(V^{\leftarrow}(a_r\xi_r + b_r)\big) - \pi_2\big(V^{\leftarrow}(b_r)\big) = \pi_2\Big(\frac{V^{\leftarrow}(a_r\xi_r + b_r)}{V^{\leftarrow}(b_r)}V^{\leftarrow}(b_r)\Big) - \pi_2\big(V^{\leftarrow}(b_r)\big).$$

Applying the definition of $\mathbf{\Pi}$-variation and (4.9) we get

$$\frac{\sigma^2(\Delta^{(r)}) - \sigma^2(b_r)}{g_2(V^{\leftarrow}(b_r))} \Rightarrow N_\Gamma. \tag{4.10}$$

Since

$$\frac{\sigma^2(b_r)}{g_2(V^{\leftarrow}(b_r))} = \frac{\pi_2(V^{\leftarrow}(b_r))}{g_2(V^{\leftarrow}(b_r))} \to \infty,$$

by (4.8), we have proved (4.5), since if we divide (4.10) by something of larger order (namely, $\sigma^2(b_r)$), we get a limit of 0. $\square$

5 Proofs of Theorems 2.1 and 2.2

In this section we first prove the conditioned limit theorem, Theorem 2.1, using both random centering and scaling; this is followed by the proof of Corollary 2.1; then we give the proof of Theorem 2.2. The proof of Theorem 2.3 is deferred to Section 6.

Proof of Theorem 2.1: Suppose X is a driftless subordinator on $(0,\infty)$ with atomless Lévy measure $\Pi(\cdot)$ on $(0,\infty)$ and its rth largest jump satisfies (1.1) for some deterministic functions $a_r > 0$ and $b_r \in \mathbb{R}$.

Conditional on $\Delta^{(r)}$, we have that ${}^{(r)}X$ is a subordinator whose Lévy measure is $\Pi_{|(0,\Delta^{(r)})}$, i.e., the measure Π restricted to $(0,\Delta^{(r)})$ (e.g., (Resnick, 1986, Prop. 2.3, p.75)).[3] So the conditional characteristic function (chf) of ${}^{(r)}X$ is

$$\mathrm{E}\Big(e^{\mathrm{i}\theta\, {}^{(r)}X}\,\Big|\,\Delta^{(r)}\Big) = \exp\Big\{\int_0^{\Delta^{(r)}} \big(e^{\mathrm{i}\theta x}-1\big)\Pi(\mathrm{d}x)\Big\},\ \theta\in\mathbb{R},$$

and the conditional chf of the centered and scaled ${}^{(r)}X$ is

$$\begin{aligned}&\mathrm{E}\Big(\exp\Big\{\mathrm{i}\theta\frac{{}^{(r)}X-\mu(\Delta^{(r)})}{\sigma(\Delta^{(r)})}\Big\}\,\Big|\,\Delta^{(r)}\Big)\\ &= \exp\Big\{\int_0^{\Delta^{(r)}}\Big(e^{\mathrm{i}\theta\frac{u}{\sigma(\Delta^{(r)})}}-1-\mathrm{i}\theta\frac{u}{\sigma(\Delta^{(r)})}\Big)\Pi(\mathrm{d}u)\Big\}. \qquad (5.1)\end{aligned}$$

Thus for (2.2) it is enough to show

$$\int_0^{\Delta^{(r)}}\Big(e^{\mathrm{i}\theta\frac{u}{\sigma(\Delta^{(r)})}}-1-\mathrm{i}\theta\frac{u}{\sigma(\Delta^{(r)})}\Big)\Pi(\mathrm{d}u)+\frac{1}{2}\theta^2 \ \to 0,\ \text{as } r\to\infty. \qquad (5.2)$$

Noting that, by (2.1),

$$\int_0^{\Delta^{(r)}}\frac{\theta^2}{2\sigma^2(\Delta^{(r)})}u^2\Pi(\mathrm{d}u)=\frac{1}{2}\theta^2,$$

and using the inequality $|e^{\mathrm{i}\theta}-1-\mathrm{i}\theta-\frac{(\mathrm{i}\theta)^2}{2}|\le|\theta|^3/3!$, $\theta\in\mathbb{R}$, the lefthand side of (5.2) is seen to be

$$\begin{aligned}&\int_0^{\Delta^{(r)}}\Big(e^{\mathrm{i}\theta\frac{u}{\sigma(\Delta^{(r)})}}-1-\mathrm{i}\theta\frac{u}{\sigma(\Delta^{(r)})}\Big)\Pi(\mathrm{d}u)-\int_0^{\Delta^{(r)}}\Big(-\frac{1}{2}\theta^2\Big)\frac{u^2}{\sigma^2(\Delta^{(r)})}\Pi(\mathrm{d}u)\\ &\le\frac{|\theta|^3}{3!}\int_0^{\Delta^{(r)}}\frac{u^3}{\sigma^3(\Delta^{(r)})}\Pi(\mathrm{d}u)\\ &\le\frac{|\theta|^3}{3!}\frac{\Delta^{(r)}}{\sigma(\Delta^{(r)})}. \qquad (5.3)\end{aligned}$$

Next we show $\Delta^{(r)}/\sigma(\Delta^{(r)})$ converges to 0 when (1.1) holds. We separate the analysis into cases according to whether the constant $\gamma<0$ or $\gamma=0$ in (1.3).

[3] Continuity of $\overline{\Pi}$ is needed to apply Prop. 2.3 of Resnick (1986); Resnick (1986) only gives the case $r=1$ but this is easily extended to $r\in\mathbb{N}$.

(i) When $\gamma < 0$, by (3.8), $\Delta^{(r)}/b_r \Rightarrow Y := \exp(-N_\Gamma|\gamma|/2)$, where N_Γ is a standard normal random variable. Furthermore, $\sigma^2(t)$ is regularly varying at 0 with index 2, so

$$\frac{\sigma^2(\Delta^{(r)})}{\sigma^2(b_r)} \to Y^2, \text{ as } r \to \infty.$$

By (4.1), it is also true that $\sigma^2(b_r) \sim \frac{1}{|\gamma|} b_r^2 \sqrt{r}$. Then we have

$$\frac{(\Delta^{(r)})^2}{\sigma^2(\Delta^{(r)})} \sim \frac{(\Delta^{(r)})^2}{b_r^2} \cdot \frac{b_r^2}{\sigma^2(b_r)} \cdot \frac{\sigma^2(b_r)}{\sigma^2(\Delta^{(r)})} = O_p(Y^2) \cdot \frac{|\gamma|}{\sqrt{r}} \cdot O_P(\frac{1}{Y^2}) \Rightarrow 0, \text{ as } r \to \infty. \tag{5.4}$$

(ii) When $\gamma = 0$, apply (3.10) and then Proposition 4.2, and we have $\Delta^{(r)}/b_r \Rightarrow 1$ and $\sigma^2(\Delta^{(r)})/\sigma^2(b_r) \Rightarrow 1$. Therefore, as in (5.4),

$$\frac{(\Delta^{(r)})^2}{\sigma^2(\Delta^{(r)})} = \frac{(\Delta^{(r)})^2}{b_r^2} \cdot \frac{b_r^2}{\sigma^2(b_r)} \cdot \frac{\sigma^2(b_r)}{\sigma^2(\Delta^{(r)})} = (1 + o_p(1)) \frac{b_r^2}{\sigma^2(b_r)} \Rightarrow 0, \text{ as } r \to \infty,$$

where the convergence to 0 follows from (4.8) for the following reason: we can use (4.4) (and recalling the definition of the function g_p in (4.4)) to write

$$\begin{aligned} \lim_{r\to\infty} \frac{\sigma^2(b_r)}{b_r^2} &= \lim_{r\to\infty} \frac{\pi_2(H(r))}{(\overline{\Pi}^{\leftarrow}(r))^2} = \lim_{t\to\infty} \frac{\pi_2(t)}{V^2(t)} = \lim_{t\to\infty} \frac{\pi_2(t)}{V^2(t)\frac{1}{2}\log t}\Big(\frac{1}{2}\log t\Big) \\ &= \lim_{t\to\infty} \frac{\pi_2(t)}{g_2(t)}\Big(\frac{1}{2}\log t\Big) = \infty. \end{aligned}$$

Thus the righthand side of (5.3) tends to 0, completing the proof of Theorem 2.1. □

Proof of Corollary 2.1: Define

$$\Phi_r^X = \frac{{}^{(r)}X - \mu(\Delta^{(r)})}{\sigma(\Delta^{(r)})}, \quad \Phi_r^\Delta = \frac{\Delta^{(r)} - b_r}{a_r},$$

and suppose f, g are non-negative continuous functions and bounded by 1. From (1.7) we have

$$\mathrm{E}g(\Phi_r^\Delta)\mathrm{E}f(N_X) \to \mathrm{E}(g(h^{\leftarrow}(N_\Gamma))\mathrm{E}f(N_X).$$

Then from (2.2) and dominated convergence

$$\mathrm{E}f(\Phi_r^X)g(\Phi_r^\Delta) = \mathrm{E}\big\{g(\Phi_r^\Delta)\mathrm{E}^{\Delta^{(r)}}\big(f(\Phi_r^X)\big)\big\} \to \mathrm{E}\{f(N_X)\}\mathrm{E}\big\{g\big(h^{\leftarrow}(N_\Gamma)\big)\big\},$$

because

$$\begin{aligned} \mathrm{E}\big\{g(\Phi_r^\Delta)\mathrm{E}^{\Delta^{(r)}}\big(f(\Phi_r^X)\big)\big\} &- \mathrm{E}g(\Phi_r^\Delta)\mathrm{E}f(N_X) \\ &= \mathrm{E}\Big(g(\Phi_r^\Delta)\mathrm{E}^{\Delta^{(r)}}\big\{f(\Phi_r^X) - \mathrm{E}f(N_X)\big\}\Big) \\ &\le \mathrm{E}\,\Big|\mathrm{E}^{\Delta^{(r)}}\big(f(\Phi_r^X)\big) - \mathrm{E}f(N_X)\Big| \to 0, \text{ as } r \to \infty. \end{aligned}$$

This completes the proof of Corollary 2.1. □

Proof of Theorem 2.2: (i) When $\gamma < 0$ we set $b_r = \overline{\Pi}^{\leftarrow}(r)$ and $a_r = |\gamma| b_r$, and then by (3.8)

$$\frac{\Delta^{(r)}}{b_r} \Rightarrow Y := e^{-N_\Gamma |\gamma|/2}, \text{ as } r \to \infty,$$

and the joint convergence in Corollary 2.1 can be written with a deterministic scaling via continuous mapping as

$$\begin{aligned}\Big(\frac{{}^{(r)}X - \mu(\Delta^{(r)})}{\sigma(b_r)}, \frac{\Delta^{(r)}}{b_r}\Big) &= \Big(\frac{{}^{(r)}X - \mu(\Delta^{(r)})}{\sigma(\Delta^{(r)})} \cdot \frac{\sigma(\Delta^{(r)})}{\sigma(b_r)}, \frac{\Delta^{(r)}}{b_r}\Big) \\ &\Rightarrow \big(N_X Y, Y\big) = \big(N_X e^{-N_\Gamma|\gamma|/2}, e^{-N_\Gamma|\gamma|/2}\big),\end{aligned}$$

where (N_X, N_Γ) are iid standard normal random variables.

Now consider the effect of changing the random centering to a deterministic one in the first component. From (4.1),

$$\sigma(b_r) = \sqrt{\sigma^2(b_r)} \sim b_r r^{1/4} \sqrt{\frac{1}{|\gamma|}}, \text{ as } r \to \infty. \tag{5.5}$$

Remember $a_r = |\gamma| b_r$ and convert (1.2) to vague convergence on $(0, \infty)$ to get

$$\frac{\Pi(b_r \mathrm{d}u)}{\sqrt{r}} \xrightarrow{v} \frac{2}{|\gamma|} \frac{\mathrm{d}u}{u}, \quad u > 0,$$

and so

$$\begin{aligned}\frac{\mu(\Delta^{(r)}) - \mu(b_r)}{b_r \sqrt{r}} &= \int_1^{\Delta^{(r)}/b_r} u \frac{\Pi(b_r \mathrm{d}u)}{\sqrt{r}} \\ &\Rightarrow \int_1^Y \frac{2}{|\gamma|} \mathrm{d}u = \frac{2}{|\gamma|}(Y - 1).\end{aligned}$$

Since $b_r \sqrt{r}/\sigma(b_r) \to \infty$ by (5.5), we have

$$\begin{aligned}\frac{{}^{(r)}X - \mu(b_r)}{b_r \sqrt{r}} &= \Big(\frac{{}^{(r)}X - \mu(\Delta^{(r)})}{b_r \sqrt{r}}\Big) + \Big(\frac{\mu(\Delta^{(r)}) - \mu(b_r)}{b_r \sqrt{r}}\Big) \\ &= o_p(1) + \Big(\frac{\mu(\Delta^{(r)}) - \mu(b_r)}{b_r \sqrt{r}}\Big) \Rightarrow \frac{2}{|\gamma|}(Y - 1), \text{ as } r \to \infty.\end{aligned}$$

(ii) When $\gamma = 0$, by Proposition 4.2, we have $\sigma(\Delta^{(r)})/\sigma(b_r) \Rightarrow 1$ with $b_r = \overline{\Pi}^{\leftarrow}(r)$. Then from Corollary 2.1,

$$\Big(\frac{{}^{(r)}X - \mu(\Delta^{(r)})}{\sigma(b_r)}, \frac{\Delta^{(r)} - b_r}{a_r}\Big) \Rightarrow \Big(N_X, \frac{N_\Gamma}{2}\Big), \tag{5.6}$$

where (N_X, N_Γ) are independent standard normal random variables. By Proposition 3.2 we may choose $a_r = 2(\overline{\Pi}^{\leftarrow}(r - \sqrt{r}) - \overline{\Pi}^{\leftarrow}(r))$. □

6 Proof of Theorem 2.3

In this section we keep $\gamma = 0$. As displayed in (5.6), we may replace the random scaling for ${}^{(r)}X$ by the deterministic scaling $\sigma(b_r)$. We investigate what happens when we try to replace $\mu(\Delta^{(r)})$ with $\mu(b_r)$ in (5.6) by a method similar to the one used in the proof of Proposition 4.2. As before we can apply (4.6), now with $p = 1$, to get

$$\mu(\Delta^{(r)}) - \mu(b_r) = \pi_1\big(V^{\leftarrow}(\Delta^{(r)})\big) - \pi_1\big(V^{\leftarrow}(b_r)\big).$$

Recall from (3.1) that we may write $(\Delta^{(r)} - b_r)/a_r = \xi_r \Rightarrow N_\Gamma/2$. The previous difference thus becomes

$$\pi_1\big(V^{\leftarrow}(a_r\xi_r + b_r)\big) - \pi_1\big(V^{\leftarrow}(b_r)\big) = \pi_1\Big(\frac{V^{\leftarrow}(a_r\xi_r + b_r)}{V^{\leftarrow}(b_r)}V^{\leftarrow}(b_r)\Big) - \pi_1\big(V^{\leftarrow}(b_r)\big).$$

Applying the definition of $\mathbf{\Pi}$-variation and (4.9) we get

$$\frac{\mu(\Delta^{(r)}) - \mu(b_r)}{g_1(V^{\leftarrow}(b_r))} \Rightarrow N_\Gamma. \tag{6.1}$$

To replace $\mu(\Delta^{(r)})$ with $\mu(b_r)$ in (5.6) requires that the difference in (6.1) be compared with $\sigma(b_r)$. The cleanest result would be if the difference were $o(\sigma(b_r))$ as $r \to \infty$, but this is not always the case and the final form of the joint limit with deterministic centering and scaling in general depends on the behaviour of the limit of

$$\lim_{r\to\infty} \frac{\sigma^2(b_r)}{g_1^2(H(r))} = \lim_{r\to\infty} \frac{\pi_2(H(r))}{g_1^2(H(r))} = \lim_{z\to\infty} \frac{\int_z^\infty \big(\overline{\Pi}^{\leftarrow}(v)\big)^2 dv}{z\big(\overline{\Pi}^{\leftarrow}(z)\big)^2}, \tag{6.2}$$

assuming there is indeed a limit. Note that the $\mathbf{\Pi}$-function $\pi_2(\cdot)$ has auxiliary function g_2 and not g_1^2 so we cannot rely on (4.8) here.

An easy example to show that $(\mu(\Delta^{(r)}) - \mu(b_r))/\sigma(b_r)$ does not always vanish is the stable subordinator from Example 1. Recall from[(i)] (3.12) we have, for $0 < \alpha < 1$,

$$\overline{\Pi}(x) = x^{-\alpha},\ x > 0; \quad \overline{\Pi}^{\leftarrow}(v) = v^{-1/\alpha},\ v > 0.$$

The ratio on the right of (6.2) is in fact constant now:

$$\frac{\int_z^\infty v^{-2/\alpha}\mathrm{d}v}{z(z^{-2/\alpha})} = \frac{\alpha}{2-\alpha}.$$

More generally, if $\overline{\Pi}(x) = x^{-\alpha}L(x)$, $x \downarrow 0$ is regularly varying at 0 with index α, then $(\overline{\Pi}^{\leftarrow}(z))^2 = z^{-2/\alpha}(L'(z))^2$, $z \to \infty$, for slowly varying functions L at 0 and L' at ∞. Then by Karamata's theorem for integrals (eg. (Bingham et al., 1989, page 27))

$$\lim_{z\to\infty} \frac{\int_z^\infty \big(\overline{\Pi}^{\leftarrow}(v)\big)^2 dv}{z\big(\overline{\Pi}^{\leftarrow}(z)\big)^2} = \frac{\alpha}{2-\alpha} = c_\alpha.$$

Since $0 < \alpha < 1$, we have $0 < c_\alpha < 1$. The converse half of Karamata's theorem ((Bingham et al., 1989, p. 30)) tells us that if

$$\lim_{z\to\infty} \frac{\int_z^\infty \big(\overline{\Pi}^{\leftarrow}(v)\big)^2 dv}{z\big(\overline{\Pi}^{\leftarrow}(z)\big)^2} = c, \text{ for some } c \in (0,\infty),$$

then $\big(\overline{\Pi}^{\leftarrow}(z)\big)^2$ is regularly varying at ∞ with index $-(c^{-1}+1)$ and $\overline{\Pi}^{\leftarrow}(z)$ is regularly varying with index $-(c^{-1}+1)/2$ at ∞. Set $1/\alpha = (c^{-1}+1)/2$. Then for $\overline{\Pi}$ to correspond to a subordinator, we need $\alpha < 1$, which makes $c < 1$.

Following this path leads to Theorem 2.3 as we state it in Section 2, giving the joint limiting distribution of ${}^{(r)}X$ and $\Delta^{(r)}$ in this particular case. Based on the technology previously developed we can now prove that theorem.

Proof of Theorem 2.3: Suppose $\gamma = 0$ and $\overline{\Pi}$ is regularly varying at 0 with index $-\alpha$, $0 < \alpha < 1$. This happens iff $\overline{\Pi}^{\leftarrow}(z)$ is regularly varying at ∞ with index $-1/\alpha = -(1+c_\alpha^{-1})/2$, where $c_\alpha = \alpha/(2-\alpha)$, or, equivalently,

$$\lim_{z\to\infty} \frac{\int_z^\infty \big(\overline{\Pi}^{\leftarrow}(v)\big)^2 dv}{\big(\overline{\Pi}^{\leftarrow}(z)\big)^2 z} = \lim_{x\to 0} \frac{\int_0^x u^2 \Pi(\mathrm{d}u)}{x^2\overline{\Pi}(x)} = c_\alpha, \text{ where } c_\alpha \in (0,1). \tag{6.3}$$

Thus, suppose (6.3) holds. By (6.2), we have $g_1(H(r))/\sigma(b_r) \to 1/\sqrt{c_\alpha}$. Then by (2.5) and (6.1),

$$\begin{aligned} \frac{{}^{(r)}X - \mu(b_r)}{\sigma(b_r)} &= \frac{{}^{(r)}X - \mu(\Delta^{(r)})}{\sigma(b_r)} + \frac{\mu(\Delta^{(r)}) - \mu(b_r)}{\sigma(b_r)} \\ &\stackrel{\mathrm{D}}{=} N_X + o_p(1) + \frac{\mu(\Delta^{(r)}) - \mu(b_r)}{g_1(H(r))} \cdot \frac{g_1(H(r))}{\sigma(b_r)} \\ &\stackrel{\mathrm{D}}{=} N_X + N_\Gamma \cdot \frac{1}{\sqrt{c_\alpha}} + o_p(1). \end{aligned} \tag{6.4}$$

Taking $r \to \infty$, this proves (2.6).

When $c_\alpha = 1$, the Lévy measure property that

$$\int_0^1 u^2 \Pi(\mathrm{d}u) = \int_1^\infty \big(\overline{\Pi}^{\leftarrow}(s)\big)^2 ds < \infty$$

(always) and the usual version of Karamata's theorem, imply that $\big(\overline{\Pi}^{\leftarrow}(x)\big)^2$ is regularly varying at infinity with index -2, so $\overline{\Pi}^{\leftarrow}(x)$ is regularly varying at infinity with index -1 and $\overline{\Pi}(x)$ is regularly varying at 0 with index -1. Conversely, if $\overline{\Pi}(x)$ is regularly varying at 0 with index -1, then (6.3) holds with $c_\alpha = 1$ and so (2.6) holds with $c_\alpha = 1$.

When (6.3) holds with $c_\alpha = 0$, then $\big(\overline{\Pi}^{\leftarrow}(z)\big)^2$ is rapidly varying at infinity ((de Haan, 1970, p. 26)) so the same is true for $\overline{\Pi}^{\leftarrow}(z)$, and, by inversion, $\overline{\Pi}(x)$ is slowly varying at 0. The converse holds as well: if $\overline{\Pi}$ is slowly varying at 0 then (6.3) holds with $c_\alpha = 0$. Referring back to (6.2) we find

$$\lim_{r\to\infty} \frac{\sigma^2(b_r)}{g_1^2(H(r))} = 0.$$

Divide on the left side of (6.4) by $g_1(H(r))$ instead of $\sigma(b_r)$. Then by (6.1) we see that (2.6) becomes

$$\Big(\frac{{}^{(r)}X-\mu(b_r)}{g_1(H(r))},\ \frac{\Delta^{(r)}-b_r}{a_r}\Big)\Rightarrow\Big(N_\Gamma,\ \frac{N_\Gamma}{2}\Big),$$

and unpacking the notation shows that $g_1\circ H(r)=b_r\sqrt{r}$ as claimed in Theorem 2.3. □

7 Final thoughts

One motivation for studying joint limit theorems as in Section 2 is to get information on limiting behaviour of ratios of a subordinator to its large jumps. See Ipsen et al. (2018) and their references for related results and applications along these lines.

There are obvious open issues we leave for another day. Restricting the investigation to subordinators clearly makes analysis easier but we would like investigate what happens if we remove the assumption that the Lévy process is non-decreasing. This would presumably require analysis of the missing case $\gamma>0$ which was necessarily absent from this paper. We also would like to investigate functional weak limit theorems for $({}^{(r)}X_t,\Delta_t^{(r)})$ as functions of t. Relevant to this, we note from Buchmann et al. (2018), Prop. 4.2, that (1.2) implies, more generally,

$$\lim_{r\to\infty}\mathrm{P}\Big(\frac{\Delta_t^{(r)}-b_{r/t}}{a_{r/t}}\le x\Big)=\mathrm{P}\big(\Delta_t^{(\infty)}\le x\big)=\Phi\big(\sqrt{t}h(x)\big),\ x\in\mathbb{R}, \tag{7.1}$$

for each $t>0$. But ${}^{(r)}X_t$ does not scale with t in the same way as $\Delta_t^{(r)}$, so generalisations of Theorems 2.1–2.3 are not straightforward in this respect. As an incidental comment we note, though it's not mentioned in Buchmann et al. (2018), that (1.2) is in fact necessary and sufficient for (1.1).

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