# Ratios of ordered points of point processes with regularly varying intensity measures 

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#### Abstract

We study limiting properties of ratios of ordered points of point processes whose intensity measures have regularly varying tails, giving a systematic treatment which points the way to "large-trimming" properties of extremal processes and a variety of applications. Our point process approach facilitates a connection with the negative binomial process of Gregoire (1984) and consequently to certain generalised versions of the Poisson-Dirichlet distribution.


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## 1. Introduction

Recent work on ratios of ordered Poisson points and ordered jumps of stable subordinators and other Lévy processes due to Kevei and Mason [13] and the present authors in [4,5], and [10] placed an emphasis on limiting properties of those ratios, and on "trimmed" versions of the process generating the points, which may have been a subordinator or a more general Lévy process.

Our aim in this paper is to give a systematic treatment of the limiting behaviour of ratios of ordered Poisson points. As is natural, we take a point process approach and make special

[^0]connection with the negative binomial process whose relevance in the present context was brought out in $[11,12]$. This connection via ratios of points enabled the construction of a generalised kind of Poisson-Dirichlet distribution which can be added to the repertoire of available models for data analytic purposes.

A related topic is the behaviour of two dimensional Poisson points ordered by the second component when the $r$ highest points are deleted. Such processes were explored in [5], and the present results provide impetus for further investigations of this kind.

The paper is structured as follows. In Section 2 we set up the point processes to be studied in Section 3 , notably a Poisson point process $\mathbb{D}$ on $\mathbb{R}^{+}=(0, \infty)$, and subsidiary point processes $\mathbb{D}_{t}^{(n)}$ and $\mathbb{D}_{t}^{(r, r+n)}$ consisting of ratios of the ordered points in $\mathbb{D}$, where the ordering is by magnitude up till a given time $t>0$, and the normalisation is by the $n$th or $(r+n)$ th largest point.

The tail of the canonical measure, $\Pi$, for the points is assumed to be regularly varying of index $-\alpha, \alpha>0$, at 0 . Under this assumption, Theorem 3.1 in Section 3 proves the weak convergence of $\mathbb{D}_{t}^{(r, r+n)}$, as $t \downarrow 0$ ("small time" convergence) to a limit comprised of a sum of independent point processes on $(0, \infty)$. The first component of the sum represents the joint limiting distribution of ratios larger than 1 of points in $\mathbb{D}$, conveniently expressed as the distribution of the order statistics of certain i.i.d. (independent and identically distributed) random variables; and the second component is a negative binomial point process, representing the limiting distribution of the (infinitely many) ratios smaller than 1 .

Further, in Section 4, we mention some interesting corollaries of Theorem 3.1, stated as separate propositions, and in Section 5 prove a converse result (Theorem 5.1) to the effect that convergence in distribution of ratios (larger or smaller than 1) implies regular (or slow, or rapid) variation of the tail of the canonical measure for points in $\mathbb{D}$.

### 1.1. Related results: order statistics of i.i.d. random variables

We conclude this section with some history relating how these kinds of Lévy process results have antecedents in the literature of order statistics of i.i.d. real-valued random variables. The general scenario there is of the order statistics $\xi_{n}^{(n)} \leq \cdots \leq \xi_{n}^{(1)}$ of i.i.d. random variables $\left(\xi_{i}\right)_{1 \leq i \leq n}$ in $\mathbb{R}$ with distribution $F$ such that $F(x)<1$ for all $x$. The asymptotic is then as $n \rightarrow \infty$ ("large time"). (In most of the results we quote, the distribution $F$ is also assumed continuous, so ties among order statistics have probability 0 . We avoid such an assumption on $\Pi$ where possible in our results but in some places it is essential, as we note later.)

An early and well-cited venture in this area was by Arov and Bobrov [1]. They considered not only the order statistics but also their sum, i.e., the random walk $S_{n}$ whose step sizes are the $\xi_{i}$, obtaining among other things results for convergence of joint distributions of deterministically normed order statistics, and, as a corollary, limiting distributions for ratios of (not necessarily successive) order statistics. This was extended to ratios of the sum after removal of a fixed number of extreme terms (the "trimmed sum") to large order statistics. (The distribution $F$ was assumed to have a density in [1].)

Smid and Stam [29] considered the $\left(\xi_{n}^{(i)}\right)_{1 \leq i \leq n}$ as above, and, in what amounts to a generalisation of and converse to one of the Arov and Bobrov [1] results, showed that $\lim _{n \rightarrow \infty} \mathrm{P}\left(\xi_{n}^{(j+1)} / \xi_{n}^{(j)} \leq x, 1 \leq j \leq k\right)=\prod_{j=1}^{k} x^{j \alpha}$ for all $x \in(0,1)$ and $k \in \mathbb{N}$ if and only if the distribution tail $\bar{F}(x) \in R V_{\infty}(-\alpha), \alpha \geq 0$. They include the $\alpha=0$ case ( $\bar{F}$ slowly varying at $\infty$ ). Their proof used Scheffé's lemma and applications of the Wiener-Tauberian theory. A converse
to another of the Arov and Bobrov [1] results is in [20]. An earlier result along the lines of Smid and Stam [29] is in [28].

Teugels [30] considered order statistics of i.i.d. random variables in the domain of attraction of a stable law, and gave results extending some of the Arov-Bobrov limit laws concerning ratios of sums of order statistics to their (trimmed) sums. For an application of these kinds of ideas in reinsurance, see Ladoucette and Teugels [15] and Fan et al. [7].

Lanzinger and Stadtmüller [16] gave a simplified version of the Smid and Stam [29] result (for the $k=1$ case) and extended this for when $F$ is in the domain of attraction of an extreme value distribution.

There is of course also a very large literature analysing various functions of order statistics of i.i.d. real-valued random variables which we do not attempt to summarise here.

Finally, in this section, we remark that while there are obvious correspondences between the (large-time) i.i.d. case and the (small time) point process case, there are significant differences too. The measure $\Pi$ is assumed to satisfy $\Pi\{(x, \infty)\}<\infty$ but $\Pi\{(0, x)\}=\infty$, for all $x>0$, so there are always infinitely many points of the process in any right neighbourhood of 0 , hence, infinitely many ordered points; whereas, in the i.i.d. case, there are of course at most $n$ order statistics in a sample of size $n$. Thus there is no immediate counterpart of some of our small time results. This feature actually simplifies some of the point process proofs, for example that of Theorem 3.1, although the formulation is more complex.

## 2. Poisson point processes and ratios of ordered points

In this section we set up the point process framework we will use. For general background on point processes, their Laplace functionals, and their convergence, etc., we refer to Resnick [25] and Daley and Vere-Jones [6].

Let $\mathbb{N}=\{1,2, \ldots\}$ and $\mathbb{N}_{0}=\{0,1,2, \ldots\}$. Suppose $\Pi$ is a Borel measure on $(0, \infty)$, locally finite at infinity. Then $\Pi$ has finite-valued tail function $\bar{\Pi}:(0, \infty) \rightarrow(0, \infty)$, defined by

$$
\bar{\Pi}(x):=\Pi\{(x, \infty)\}, \quad x>0
$$

a right-continuous, non-increasing function. Let

$$
\bar{\Pi} \leftarrow(x)=\inf \{y>0: \bar{\Pi}(y) \leq x\}, x>0
$$

be the right-continuous inverse of $\bar{\Pi}$. Assume throughout that $\Pi\{(0, \infty)\}=\bar{\Pi}(0+)=\infty$.
With $\delta_{\{x\}}$ denoting a point mass at $x \in(0, \infty)$, let $\mathbb{D}=\sum_{s>0} \delta_{\left\{s, \Delta_{s}\right\}}$ define a Poisson point process on $[0, \infty) \times(0, \infty)$ with intensity measure $\mathrm{d} s \times \Pi(\mathrm{d} x)$. Since $\bar{\Pi}(0+)=\infty$, there are infinitely many non-zero points of $\mathbb{D}$ in any right neighbourhood of 0 . Write those points of $\mathbb{D}$ occurring in $[0, t], t>0$, in decreasing order of magnitude, possibly with ties, as;

$$
\infty>\Delta_{t}^{(1)} \geq \cdots \geq \Delta_{t}^{(r)} \geq \cdots>0
$$

then for each $t>0$ define a Poisson point process on $(0, \infty)$ with intensity measure $t \Pi(\mathrm{~d} x)$ by

$$
\begin{equation*}
\mathbb{D}_{t}=\sum_{j \geq 1} \delta_{\left\{\Delta_{t}^{(j)}\right\}} \tag{2.1}
\end{equation*}
$$

The jump process $\left(\Delta X_{t}:=X_{t}-X_{t-}\right)_{t>0}$ of a real-valued Lévy process $\left(X_{t}\right)_{t \geq 0}$ forms a Poisson point process on $(0, \infty)$ whose intensity measure is the canonical measure of $X$. A representation detailed in Buchmann, Fan and Maller [4] shows how to construct all processes $\mathbb{D}_{t}$ as in (2.1) on the same space when the $\Delta_{t}^{(j)}$ are the ordered jumps of the jump process of
a Lévy process. Their construction applies to our present, quite general, setup (not restricted to Lévy processes), and we can carry their formulations over directly, as follows.

Since $\bar{\Pi}(0+)=\infty$, all $\Delta_{t}^{(j)}$ are positive a.s. but $\Delta_{t}^{(j)} \downarrow 0$ a.s. as $t \downarrow 0$ for each $j \in \mathbb{N}$. Let $\left(\mathfrak{E}_{i}\right)$ be an i.i.d. sequence of exponentially distributed random variables with common parameter $\mathrm{E}_{i}=1$. Then $\Gamma_{j}:=\sum_{i=1}^{j} \mathfrak{E}_{i}$ is a $\operatorname{Gamma}(j, 1)$ random variable, $j \in \mathbb{N}$, and $\left\{\Gamma_{j}, j \geq 1\right\}$ can be regarded as the points of a homogeneous unit rate Poisson process on $\mathbb{R}^{+}$written as

$$
\begin{equation*}
\sum_{j \geq 1} \delta_{\left\{\Gamma_{j}\right\}} \tag{2.2}
\end{equation*}
$$

Then the Buchmann, Fan and Maller [4] representation is given by setting

$$
\begin{equation*}
\left\{\Delta_{t}^{(j)}\right\}_{j \geq 1} \stackrel{\mathrm{D}}{=}\left\{\bar{\Pi}^{\leftarrow}\left(\Gamma_{j} / t\right)\right\}_{j \geq 1}, t>0 \tag{2.3}
\end{equation*}
$$

For earlier and related representations like this one can consult [17,18,19]; [27], pp. 21, 30; [25], Ex. 3.38, p. 139; [24], Sect. 2.4; and [8].

We will write the gamma density in the form

$$
\begin{equation*}
\mathrm{P}\left(\Gamma_{j} \in \mathrm{~d} x\right)=\frac{x^{j-1} e^{-x} \mathrm{~d} x}{\Gamma(j)} \mathbf{1}_{\{x>0\}}, j \in \mathbb{N}, \tag{2.4}
\end{equation*}
$$

for the density of $\Gamma_{j}$, which should not be confused with the Gamma function, $\Gamma(a)=$ $\int_{0}^{\infty} x^{a-1} e^{-x} \mathrm{~d} x, a>0$. Recall that a beta random variable $\mathrm{B}_{a, b}$ on $(0,1)$ with parameters $a, b>0$ has density function

$$
f_{B}(x)=\frac{\Gamma(a+b)}{\Gamma(a) \Gamma(b)} x^{a-1}(1-x)^{b-1}=\frac{1}{B(a, b)} x^{a-1}(1-x)^{b-1}, 0<x<1,
$$

where $B(a, b):=\Gamma(a) \Gamma(b) / \Gamma(a+b)$. Thus

$$
\begin{equation*}
\mathrm{P}\left(\mathrm{~B}_{a, b} \leq x\right)=\frac{1}{B(a, b)} \int_{0}^{x} y^{a-1}(1-y)^{b-1} \mathrm{~d} y=: B(a, b ; x), 0<x<1 \tag{2.5}
\end{equation*}
$$

where $B(a, b ; x)$ is the incomplete Beta function.
To complete the setting up, for the conditioning arguments in Section 4 we require the next result. Recall the Poisson points $\left\{\Gamma_{j}, j \geq 1\right\}$ described before (2.2).

Lemma 2.1. Suppose $\left\{\Gamma_{n}, n \geq 1\right\}$ are successive homogeneous Poisson points obtained by summing i.i.d. unit exponential random variables. Then for any $n \geq 1$,

$$
\left(\frac{\Gamma_{2}}{\Gamma_{1}}, \ldots, \frac{\Gamma_{n+1}}{\Gamma_{n}}, \Gamma_{n+1}\right)
$$

are independent random variables, with $\Gamma_{r} / \Gamma_{r+n} \stackrel{\mathrm{D}}{=} B_{r, n}$.
Proof of Lemma 2.1. We use the standard fact (e.g., [26, Lemma 4.5.1(b), p. 322]) that the conditional density of $\left(\Gamma_{1}, \ldots, \Gamma_{n}\right)$ given $\Gamma_{n+1}=s>0$ is the same as the joint density of the order statistics $s U_{n}^{(n)} \leq \cdots \leq s U_{n}^{(1)}$, where $U_{n}^{(j)}$ is the $j$ th largest of a uniform sample of size $n$ on $[0,1]$. Recall also that the spacings from a unit exponential sample of size $n$,
$\left(E_{n}^{(j+1)}-E_{n}^{(j)}\right)_{j=0, \ldots, n-1}$, are independent and satisfy $U_{n}^{(j)} \stackrel{\mathrm{D}}{=} \exp \left(-E_{n}^{(j)}\right)$. Thus, with obvious notation,

$$
\begin{align*}
\left(\frac{\Gamma_{2}}{\Gamma_{1}}, \ldots, \left.\frac{\Gamma_{n+1}}{\Gamma_{n}} \right\rvert\, \Gamma_{n+1}=s\right) & \stackrel{\mathrm{D}}{=}\left(\frac{s U_{n}^{(n-1)}}{s U_{n}^{(n)}}, \ldots, \frac{s U_{n}^{(3)}}{s U_{n}^{(2)}}, \frac{s}{s U_{n}^{(1)}}\right) \\
& =\left(\frac{U_{n}^{(n-1)}}{U_{n}^{(n)}}, \ldots, \frac{U_{n}^{(3)}}{U_{n}^{(2)}}, \frac{1}{U_{n}^{(1)}}\right) . \tag{2.6}
\end{align*}
$$

Switching to exponential order statistics gives this equal in distribution to

$$
\left(e^{E_{n}^{(n)}-E_{n}^{(n-1)}}, \ldots, e^{E_{n}^{(1)}}\right)
$$

The claimed result follows from exponential spacings being independent and from the fact that the last expression in (2.6) does not depend on $s$. That $\Gamma_{r} / \Gamma_{r+n} \stackrel{\mathrm{D}}{=} B_{r, n}$ is well known from "beta-gamma algebra" (e.g., [21, p. 11]).

We are interested in the convergence behaviour of ratios of the order statistics $\Delta_{t}^{(j)}$, as $t \downarrow 0$. The basic assumption is the regular variation of the tail function $\bar{\Pi}(x)$. Write $R V_{0}(\beta)$ (resp. $R V_{\infty}(\beta)$ ) for the real-valued functions regularly varying at 0 (resp, infinity) with index $\beta$. We have $\bar{\Pi}(x) \in R V_{0 / \infty}(-\alpha), 0 \leq \alpha \leq \infty$, if and only if

$$
\lim _{x \rightarrow 0 / \infty} \frac{\bar{\Pi}(x \lambda)}{\bar{\Pi}(x)}=\lambda^{-\alpha}, \text { for } \lambda>0
$$

Here we interpret $\lambda^{-\infty}=0 \cdot \mathbf{1}_{\{\lambda>1\}}+1 \cdot \mathbf{1}_{\{\lambda=1\}}+\infty \cdot \mathbf{1}_{\{\lambda<1\}}$ and $1 / 0 \equiv \infty$. From [3, p. 28-29] we know that $\bar{\Pi}(x) \in R V_{0 / \infty}(-\alpha)$ iff $\bar{\Pi} \leftarrow(x) \in R V_{\infty / 0}(-1 / \alpha)$. The slowly varying functions at 0 or $\infty$ are denoted by $R V_{0 / \infty}(0)$, and $R V_{0 / \infty}(\infty)$ are the rapidly varying functions at 0 or $\infty$.

When $\bar{\Pi}(\cdot) \in R V_{0}(-\alpha)$ with $0 \leq \alpha \leq \infty$ or, equivalently, $\bar{\Pi} \leftarrow(\cdot) \in R V_{\infty}(-1 / \alpha)$, we have the easily verified convergence (with the interpretation as above when $\alpha=0$ or $\alpha=\infty$ ):

$$
\begin{equation*}
t \bar{\Pi}(u \bar{\Pi} \leftarrow(y / t)) \sim \frac{\bar{\Pi}\left(u y^{-1 / \alpha} \bar{\Pi}^{\leftarrow}(1 / t)\right)}{\bar{\Pi}\left(\bar{\Pi}^{\leftarrow}(1 / t)\right)} \rightarrow u^{-\alpha} y \text { as } t \downarrow 0 \text {, for all } u, y>0 \text {. } \tag{2.7}
\end{equation*}
$$

When $0<\alpha<2$ we can interpret the $\Gamma_{j}^{-1 / \alpha}$ used in (2.2) as the $j$ th largest jump of a stable process $\left(S_{t}\right)_{0<t \leq 1}$ with Lévy measure $\Lambda(\mathrm{d} x)=\alpha x^{-\alpha-1} \mathbf{1}_{\{x>0\}}$; but we allow any $\alpha>0$. The process $\mathbb{S}:=\sum_{j \geq 1} \delta_{\left\{\Gamma_{j}^{-1 / \alpha}\right\}}$ is a Poisson point process on $(0, \infty)$ with intensity measure $\Lambda$.

## 3. Ratios of ordered points

In this section we give a general result for the convergence of point processes of ratios of ordered points of $\mathbb{D}_{t}$. Fix $r \in \mathbb{N}_{0}, n \in \mathbb{N}$ and $t>0$. Define the point processes on $(0, \infty)$ :

$$
\begin{equation*}
\mathbb{D}_{t}^{(n)}:=\sum_{j \geq n+1} \delta_{\left\{\Delta_{t}^{(j)} / \Delta_{t}^{(n)}\right\}} \tag{3.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathbb{D}_{t}^{(r, r+n)}:=\sum_{j \geq r+1} \delta_{\left\{\Delta_{t}^{(j)} / \Delta_{t}^{(r+n)}\right\}}=\sum_{j=1}^{n-1} \delta_{\left\{\Delta_{t}^{(r+j)} / \Delta_{t}^{(r+n)}\right\}}+1+\mathbb{D}_{t}^{(r+n)} \tag{3.2}
\end{equation*}
$$

Conditionally on $\left\{\Delta_{t}^{(n)}=z\right\}, z>0$, the points $\left(\Delta_{t}^{(j)}\right)_{j \geq n+1}$ comprise a Poisson point process with intensity measure $t \Pi$ restricted to $(0, z)$. Thus, the Laplace functional of $\mathbb{D}_{t}^{(n)}$, conditional
on $\left\{\Delta_{t}^{(n)}=z\right\}$, is

$$
\begin{align*}
\mathrm{E}\left(e^{-\mathbb{D}_{t}^{(n)}(f)} \mid \Delta_{t}^{(n)}=z\right) & =\mathrm{E}\left(\exp \left(-\int_{0<x<1} f(x) \mathbb{D}_{t}^{(n)}(\mathrm{d} x)\right) \mid \Delta_{t}^{(n)}=z\right) \\
& =\exp \left(-t \int_{0<x<1}\left(1-e^{-f(x)}\right) \Pi(z \mathrm{~d} x)\right) \tag{3.3}
\end{align*}
$$

where $f \in \mathcal{F}_{+}$, the nonnegative measurable functions on $\mathbb{R}^{+}$.
Analogous to (3.1), define

$$
\begin{equation*}
\mathbb{B}^{(n)}=\sum_{j \geq n+1} \delta_{\left\{\left(\Gamma_{j} / \Gamma_{n}\right)^{-1 / \alpha}\right\}}, n \in \mathbb{N} . \tag{3.4}
\end{equation*}
$$

The point process in (3.4) has Laplace functional at $f$ equal to

$$
\begin{equation*}
\mathrm{E}\left(e^{-\mathbb{B}^{(n)}(f)}\right)=\left(1+\int_{0}^{1}\left(1-e^{-f(x)}\right) \Lambda(\mathrm{d} x)\right)^{-n} \tag{3.5}
\end{equation*}
$$

Eq. (3.5) identifies $\mathbb{B}^{(n)}$ as a negative binomial point process with base measure $\Lambda^{*}(\mathrm{~d} x):=$ $\Lambda(\mathrm{d} x) \mathbf{1}_{\{0<x<1\}}$, denoted by $\mathcal{B N}\left(n, \Lambda^{*}\right)$, in the notation of Gregoire [9]. (See (3.13) for the proof of (3.5).)

The next theorem shows the weak convergence (denoted by ' $\xrightarrow{\mathrm{D}}$ ') of $\mathbb{D}_{t}^{(r, r+n)}$ as $t \downarrow 0$ to a limit comprised of independent components of $\mathbb{B}^{(r+n)}$ and a mixture of beta random variables. In this theorem we keep $\alpha$ positive and finite.

Theorem 3.1. Assume $\bar{\Pi}(\cdot) \in R V_{0}(-\alpha), 0<\alpha<\infty, f \in \mathcal{F}_{+}$, and $n, r \in \mathbb{N}$. Then
(i) In the space of point measures $M_{p}(0, \infty)$ with the vague topology (cf. [25], Ch. 3)

$$
\begin{equation*}
\mathbb{D}_{t}^{(r, r+n)} \xrightarrow{\mathrm{D}} \sum_{j=1}^{\infty} \delta_{\left\{\left(\Gamma_{r+j} / \Gamma_{r+n}\right)^{-1 / \alpha}\right\}}=\sum_{j=1}^{n-1} \delta_{\left\{\left(\Gamma_{r+j} / \Gamma_{r+n}\right)^{-1 / \alpha}\right\}}+\delta_{\{1\}}+\mathbb{B}^{(r+n)}, \tag{3.6}
\end{equation*}
$$

as $t \downarrow 0$. The limit in (3.6) has Laplace functional at $f$ equal to

$$
\begin{equation*}
\mathrm{E}\left(e^{-f\left(J\left(B_{r, n}^{1 / \alpha}\right)\right)}\right)^{n-1} e^{-f(1)} \mathrm{E}\left(e^{-\mathbb{B}^{(r+n)}(f)}\right) \tag{3.7}
\end{equation*}
$$

where, for each $u \in(0,1), J(u)$ has distribution

$$
\begin{equation*}
\mathrm{P}(J(u) \in \mathrm{d} x)=\frac{\Lambda(\mathrm{d} x) \mathbf{1}_{\{1<x<1 / u\}}}{1-u^{\alpha}}, x>0 \tag{3.8}
\end{equation*}
$$

$B_{r, n}$ is a $\operatorname{Beta}(r, n)$ random variable independent of $J(u)$, and the third factor on the right of (3.7) is determined from (3.5).
(ii) For $r=0$,

$$
\begin{equation*}
\lim _{t \downarrow 0} \mathrm{E}\left(e^{-\mathbb{D}_{t}^{(0, n)}(f)}\right)=\mathrm{E}\left(e^{-f(L)}\right)^{n-1} e^{-f(1)} \mathrm{E}\left(e^{-\mathbb{B}^{(n)}(f)}\right) \tag{3.9}
\end{equation*}
$$

where $L$ is a random variable with distribution

$$
\begin{equation*}
\mathrm{P}(L \in \mathrm{~d} x)=\Lambda(\mathrm{d} x) \mathbf{1}_{\{x>1\}} . \tag{3.10}
\end{equation*}
$$

Proof of Theorem 3.1. (i) Using the representation in (2.3) and the fact that $\bar{\Pi} \leftarrow \in R V_{\infty}(-1 / \alpha)$, we immediately get, as $t \downarrow 0$, with almost sure convergence,

$$
\begin{equation*}
\left(\frac{\Delta_{t}^{(r+j)}}{\Delta_{t}^{(r+n)}} ; j \geq 1\right) \stackrel{\mathrm{D}}{=}\left(\frac{\bar{\Pi} \leftarrow\left(\Gamma_{r+j} / t\right)}{\bar{\Pi}^{\leftarrow}\left(\Gamma_{r+n} / t\right)} ; j \geq 1\right) \rightarrow\left(\left(\frac{\Gamma_{r+j}}{\Gamma_{r+n}}\right)^{-1 / \alpha} ; j \geq 1\right) \tag{3.11}
\end{equation*}
$$

for each $r \in \mathbb{N}_{0}, n \in \mathbb{N}$. Analogous to (2.2), the left hand side of (3.11) equivalently defines $\mathbb{D}_{t}^{(r, r+n)}$, and so the convergence in (3.11) establishes the convergence in (3.6). By separating the ratios in the limit process into those bigger than 1 , equal to 1 , or smaller than 1 , and recalling (3.4), we get the form on the right hand side of (3.6).

The points in the two limit point processes in (3.6) occur in non-overlapping regions, so, conditionally on $\Gamma_{r+n}$, they are independent of each other. In fact (see Lemma 2.1), $\Gamma_{r} / \Gamma_{r+n} \stackrel{\mathrm{D}}{=} B_{r, n}$ with $B_{r, n}$ independent of $\Gamma_{r+n}$, so the components on the right hand side of (3.6) are also unconditionally independent. Thus the Laplace transform can be given in the product form of (3.7).

Next we will derive the Laplace functional for each component separately. For ratios bigger than 1, we can write, for $f \in \mathcal{F}_{+}$,

$$
\begin{gather*}
\operatorname{Eexp}\left(\left.-\sum_{j=1}^{n-1} f\left(\left(\frac{\Gamma_{r+j}}{\Gamma_{r+n}}\right)^{-1 / \alpha}\right) \right\rvert\, B_{r, n}=s\right)=\mathrm{E} \exp \left(-\sum_{j=1}^{n-1} f\left(\left((1-s) U_{j}+s\right)^{-1 / \alpha}\right)\right) \\
=\left(\int_{0}^{1} \exp \left(-f\left(((1-s) u+s)^{-1 / \alpha}\right)\right) \mathrm{d} u\right)^{n-1} . \tag{3.12}
\end{gather*}
$$

Here we used the same argument as in the proof of Lemma 2.1 to replace $\Gamma_{r+j} / \Gamma_{r+n}$ by $(1-s) U_{j}+s, 1 \leq j \leq n-1$. Setting $y=((1-s) u+s)^{-1 / \alpha}$, (3.12) is seen to be equal to

$$
\left(\int_{1}^{s^{-1 / \alpha}} e^{-f(y)} \alpha y^{-\alpha-1} \frac{\mathrm{~d} y}{1-s}\right)^{n-1}
$$

Take expectations in (3.12) to get

$$
\operatorname{Eexp}\left(-\sum_{j=1}^{n-1} f\left(\left(\frac{\Gamma_{r+j}}{\Gamma_{r+n}}\right)^{-1 / \alpha}\right)\right)=\mathrm{E}\left(\int_{1}^{B_{r, n}^{-1 / \alpha}} e^{-f(y)} \alpha y^{-\alpha-1} \frac{\mathrm{~d} y}{1-B_{r, n}}\right)^{n-1}
$$

Conditional on $B_{r, n}^{1 / \alpha}$, this is the Laplace functional of $n-1$ i.i.d. random variables with the distribution in (3.8).

Next we compute the intensity measure of the limit point process with ratios less than 1 , that is, the process $\mathbb{D}_{t}^{(r+n)}=\sum_{j \geq n+1} \delta_{\left\{\left(\Gamma_{r+j} / \Gamma_{r+n}\right)^{-1 / \alpha}\right\}}$ in (3.2). Conditional on $\Gamma_{r+n}$, the process $\sum_{j \geq n+1} \delta_{\left\{\Gamma_{r+j} / \Gamma_{r+n}\right\}}$ is a Poisson process with mean measure $\Gamma_{r+n} \mathrm{~d} x$, where $\mathrm{d} x$ is the Lebesgue measure. Then the image measure of $\Gamma_{r+n} \mathrm{~d} x$ under the map $T: x \mapsto x^{-1 / \alpha}$ is $\Gamma_{r+n} \Lambda(\mathrm{~d} x)$. Hence, for any $f \in \mathcal{F}_{+}$,

$$
\begin{align*}
& \operatorname{Eexp}\left(-\sum_{j \geq n+1} f\left(\left(\frac{\Gamma_{r+j}}{\Gamma_{r+n}}\right)^{-1 / \alpha}\right)\right) \\
& =\int_{y>0} \exp \left(-\int_{0<x<1}\left(1-e^{-f(x)}\right) y \Lambda(\mathrm{~d} x)\right) \mathrm{P}\left(\Gamma_{r+n} \in \mathrm{~d} y\right) \\
& =\left(1+\int_{0}^{1}\left(1-e^{-f(x)}\right) \Lambda(\mathrm{d} x)\right)^{-r-n} \tag{3.13}
\end{align*}
$$

Referring to (3.5), this is the Laplace transform of a negative binomial point process $\mathcal{B N}(r+$ $\left.n, \Lambda(\mathrm{~d} x) \mathbf{1}_{\{0<x<1\}}\right)$ at $f$. (Note that (3.13) also establishes that $\mathbb{B}^{(n)}$ has the Laplace functional in (3.5).)
(ii) $(r=0)$ The proof of (3.9) is very similar. The treatment for ratios smaller than or equal to 1 is exactly the same. For points bigger than 1 , no conditioning is necessary. We omit further details.

[^1]Remark 3.1. The first component on the RHS of the limit in (3.7) shows that, after deleting the $r$ largest points, the sum $\sum_{i=r+1}^{r+n-1}\left(\Gamma_{i} / \Gamma_{r+n}\right)^{-1 / \alpha}$ has the distribution of a sum of i.i.d. random variables, once we condition on a $B_{r, n}^{1 / \alpha}$ random variable. The third component on the RHS of (3.7) is the negative binomial point process $\mathbb{B}^{(r+n)}$ with base measure $\Lambda^{*}$. So we have the nice representation resulting from the decomposition of the original process into parts including ratios smaller than 1 and greater than 1.

Ratios of jumps of stable subordinators also featured prominently in the work of Pitman and Yor [23]. Much subsequent related research involved Poisson-Dirichlet distributions and their involvement in fragmentation and coalescence problems; see [2] and references therein. An early influential paper was [14]. The resulting processes have found wide application in a variety of applied areas ranging from Bayesian statistics to models for species diversity; see for example the list in [22, Sect. 1].

When $0<\alpha<1, r=0$, similar results to some of those in Theorem 3.1 were obtained in Lemma 24 of Pitman and Yor [23] but without explicit reference being made to the negative binomial point process of Gregoire [9]. Our result allows the bigger range of $\alpha, \alpha>0$, and generalises to point processes with intensity measures whose tails are regularly varying, rather than dealing only with jumps of subordinators. In general, in our scenario, the points of the limiting process may not be summable. As a special case, for example, we deal elsewhere (in [10]) with Lévy processes in the domain of attraction of a stable process with index $\alpha \in(1,2)$; compensating the process is then essential.

In the next section we draw out some ramifications of Theorem 3.1.

## 4. Corollaries, special cases and further results

Theorem 3.1 is expressed as the convergence of point processes. In this section we express the theorem in a different form in order to facilitate comparisons with earlier results in the literature; we also extend the result to the $\alpha=0$ or $\alpha=+\infty$ cases and consider limits of conditional distributions.

The discussion is again conveniently divided into parts covering ratios smaller than or greater than 1 . Recall $\mathbb{D}_{t}^{(n)}$ defined in (3.1) and define the ratio

$$
\begin{equation*}
W_{r, n}(t):=\frac{\Delta_{t}^{(r+n)}}{\Delta_{t}^{(r)}}, r, n \in \mathbb{N}, t>0 \tag{4.1}
\end{equation*}
$$

Proposition 4.1 (Ratios Smaller than 1). Assume $\bar{\Pi}(\cdot) \in R V_{0}(-\alpha), 0 \leq \alpha \leq \infty$.
(i) Suppose $0<\alpha<\infty$. Then, for each $n \in \mathbb{N}$,

$$
\begin{equation*}
\mathbb{D}_{t}^{(n)} \xrightarrow{\mathrm{D}} \mathbb{B}^{(n)} \text {, as } t \downarrow 0, \tag{4.2}
\end{equation*}
$$

where $\mathbb{B}^{(n)}$ is distributed as $\mathcal{B N}\left(n, \Lambda^{*}\right)$ with the Laplace functional in (3.5).
(ii) As $t \downarrow 0$, for each $r, n \in \mathbb{N}$,

$$
\begin{equation*}
\left(\frac{\Delta_{t}^{(r+1)}}{\Delta_{t}^{(r)}}, \ldots, \frac{\Delta_{t}^{(r+n)}}{\Delta_{t}^{(r+n-1)}}\right) \xrightarrow{\mathrm{D}}\left(Y_{r}, \ldots, Y_{r+n-1}\right), \tag{4.3}
\end{equation*}
$$

where $Y_{k}, k \in \mathbb{N}$, are mutually independent random variables with Beta $(k \alpha, 1)$ distributions. When $\alpha=0$ or $\alpha=\infty$, (4.3) remains true with each $Y_{k}$ equal to 0 or with each $Y_{k}$ equal to 1 , respectively.
(iii) When $0<\alpha<\infty, W_{r, n}(t)$ in (4.1) has limiting distribution as $t \downarrow 0$ that of

$$
\begin{equation*}
\prod_{i=1}^{n} Y_{r+i-1} \stackrel{\mathrm{D}}{=}\left(\frac{\Gamma_{r}}{\Gamma_{r+n}}\right)^{1 / \alpha} \stackrel{\mathrm{D}}{=} B_{r, n}^{1 / \alpha}=: W_{r, n} \tag{4.4}
\end{equation*}
$$

where the $Y_{i}$ are as in (4.3) and $W_{r, n}$ has density

$$
\begin{equation*}
f_{W_{r, n}}(w)=\frac{\left(1-w^{\alpha}\right)^{n-1} \alpha w^{\alpha r-1}}{B(r, n)}, 0<w<1 . \tag{4.5}
\end{equation*}
$$

Proof of Proposition 4.1. (i) The convergence in (4.2) is an immediate consequence of (3.9).
(ii) When $0<\alpha<\infty$ the convergence in (4.3) follows immediately from (3.11) and Lemma 2.1. When $\alpha=0$ or $\alpha=\infty$, (3.11) remains true with the appropriate interpretations as outlined in the discussion leading to (2.7).
(iii) Eq. (4.4) is implied by (4.3) and the density in (4.5) is easily calculated.

Remark 4.1. Treated as ratios of ordered jumps of a subordinator, Kevei and Mason [13] proved the case $n=1$ in (4.3), among other results comparing the magnitudes of ordered jumps of a subordinator with the magnitude of the subordinator itself. Proposition 4.1 is a multidimensional version of their Theorem 1.2, with the $\left(\Delta_{t}\right)$ treated as points in $\mathbb{D}$, in their own right. (Kevei and Mason [13] also proved converse results; see our Section 5.)

Next we prove some results concerning convergence of certain conditional probabilities. For these, we assume that $\Pi$ is atomless, or, equivalently, $\bar{\Pi}(x)$ is continuous in $x$. After the proof of the proposition we explain why this requirement is needed. In what follows, let $\sigma(X)$ denote the $\sigma$-field generated by a random element $X$ and let $\sigma\left\{B_{\kappa}, \kappa \in \mathcal{K}\right\}$ be the $\sigma$-field generated by events indexed by some index set $\mathcal{K}$. When $\bar{\Pi}(x)$ is continuous for all $x>0$, we have for any $i \in \mathbb{N}$

$$
\begin{equation*}
\sigma\left(\Delta_{t}^{(i)}\right)=\sigma\left(\bar{\Pi} \leftarrow\left(\Gamma_{i} / t\right)\right)=\sigma\left\{\Gamma_{i} \leq t \bar{\Pi}(x), x>0\right\}=\sigma\left\{\Gamma_{i} \leq y, y>0\right\}=\sigma\left(\Gamma_{i}\right) \tag{4.6}
\end{equation*}
$$

Proposition 4.2 (Ratios Greater than 1). Assume $\bar{\Pi}(\cdot) \in R V_{0}(-\alpha)$ with $0 \leq \alpha \leq \infty$ and is continuous. Take $x_{k} \geq 1$ for $0 \leq k \leq n-1, n=2,3, \ldots, r \in \mathbb{N}$ and $x>0$.
(a) Suppose $0<\alpha<\infty$.
(i) Then, for $0<u<1$,

$$
\begin{align*}
& \lim _{t \downarrow 0} \mathrm{P}\left(\frac{\Delta_{t}^{(r+k)}}{\Delta_{t}^{(r+n)}}>x_{k}, 0 \leq k \leq n-1 \mid W_{r, n}(t)=u, \Delta_{t}^{(r+n)}=x\right) \\
& =\mathbf{1}_{\left\{u<x_{0}^{-1}\right\}} \mathrm{P}\left(J_{n-1}^{(k)}(u)>x_{k}, 1 \leq k \leq n-1\right), \tag{4.7}
\end{align*}
$$

where $J_{n-1}^{(1)}(u) \geq J_{n-1}^{(2)}(u) \ldots \geq J_{n-1}^{(n-1)}(u)$ are distributed like the decreasing order statistics of $n-1$ i.i.d. random variables $\left(J_{i}(u)\right)_{1 \leq i \leq n-1}$, each having the distribution in (3.8).
(ii) For n, $r, x_{k}, x$ as specified,

$$
\begin{align*}
& \lim _{t \downarrow 0} \mathrm{P}\left(\frac{\Delta_{t}^{(r+k)}}{\Delta_{t}^{(r+n)}}>x_{k}, 0 \leq k \leq n-1 \mid \Delta_{t}^{(r+n)}=x\right) \\
& =\mathrm{P}\left(\mathrm{~B}_{r, n}^{1 / \alpha} \leq x_{0}^{-1}, J_{n-1}^{(k)}\left(\mathrm{B}_{r, n}^{1 / \alpha}\right)>x_{k}, 1 \leq k \leq n-1\right) \tag{4.8}
\end{align*}
$$

where the $J_{i}(u)$ are as in (4.7) and $B_{r, n}$ is a $\operatorname{Beta}(r, n)$ random variable independent of $\left(J_{i}(u)\right)_{1 \leq i \leq n-1}$.
(b) When $\alpha=0$, each ratio $\Delta_{t}^{(r+k)} / \Delta_{t}^{(r+n)} \xrightarrow{\mathrm{P}} \infty$ as $t \downarrow 0$, for $1 \leq k \leq n-1$. When $\alpha=\infty$, each ratio $\Delta_{t}^{(r+k)} / \Delta_{t}^{(r+n)} \xrightarrow{\mathrm{P}} 1$ ast $\downarrow 0$, for $1 \leq k \leq n-1$.

Proof of Proposition 4.2. (a) (i) Let $f: \mathbb{R}_{+}^{n} \mapsto \mathbb{R}_{+}$be bounded and continuous and consider

$$
\begin{align*}
& \mathrm{E}\left(\left.f\left(\frac{\Delta_{t}^{(r+k)}}{\Delta_{t}^{(r+n)}}, 0 \leq k \leq n-1\right) \right\rvert\, \frac{\Delta_{t}^{(r+n)}}{\Delta_{t}^{(r)}}, \Delta_{t}^{(r+n)}\right) \\
& =\mathrm{E}\left(\left.f\left(\frac{\bar{\Pi} \leftarrow\left(t^{-1} \Gamma_{r+n} \cdot \frac{\Gamma_{r+k}}{\Gamma_{r+n}}\right)}{\bar{\Pi} \leftarrow\left(t^{-1} \Gamma_{r+n}\right)}, 0 \leq k \leq n-1\right) \right\rvert\, \frac{\bar{\Pi} \leftarrow\left(\Gamma_{r+n} / t\right)}{\bar{\Pi}^{\leftarrow}\left(\Gamma_{r} / t\right)}, \Gamma_{r+n}\right) . \tag{4.9}
\end{align*}
$$

Now, keeping in mind (4.6), we have the equality of $\sigma$-fields

$$
\sigma\left(\frac{\bar{\Pi} \leftarrow\left(\Gamma_{r+n} / t\right)}{\bar{\Pi}^{\leftarrow}\left(\Gamma_{r} / t\right)}, \Gamma_{r+n}\right)=\sigma\left(\bar{\Pi} \leftarrow\left(\Gamma_{r} / t\right), \Gamma_{r+n}\right)=\sigma\left(\Gamma_{r}, \Gamma_{r+n}\right)=\sigma\left(\Gamma_{r} / \Gamma_{r+n}, \Gamma_{r+n}\right) .
$$

The conditional expectation in (4.9) is therefore of the form

$$
\mathrm{E}\left(f_{t}\left(X_{0}, Z_{0}, \ldots, Z_{n-1}\right) \mid X_{0}, Z_{0}\right)
$$

where by Lemma 2.1, with $r+n$ replacing $n+1$, the random variables $X_{0}=\Gamma_{r+n}$ and the vector $\left(Z_{k}=\Gamma_{r+k} / \Gamma_{r+n}, k=0, \ldots, n-1\right)$ are independent of each other. Note $Z_{0}=B_{r, n}$. By the regular variation of $\bar{\Pi} \leftarrow$ at 0 , as $t \downarrow 0$,

$$
\begin{aligned}
f_{t}\left(X_{0}, Z_{0}, \ldots, Z_{n-1}\right) & \rightarrow f\left(\left(\Gamma_{r+k} / \Gamma_{r+n}\right)^{-1 / \alpha}, 0 \leq k \leq n-1\right) \\
& =f\left(Z_{k}^{-1 / \alpha}, 0 \leq k \leq n-1\right)
\end{aligned}
$$

so by dominated convergence the conditional expectation in (4.9) converges to

$$
\mathrm{E}\left(f\left(Z_{k}^{-1 / \alpha}, 0 \leq k \leq n-1\right) \mid X_{0}, Z_{0}\right)=\mathrm{E}\left(f\left(Z_{k}^{-1 / \alpha}, 0 \leq k \leq n-1\right) \mid Z_{0}\right)
$$

where the last equality holds since $X_{0}$ is independent of the $Z$ 's.
Recall the discussion around (3.12), which shows that, conditionally on $Z_{0},\left(Z_{1}^{-1 / \alpha}, \ldots\right.$, $\left.Z_{n-1}^{-1 / \alpha}\right)$ has joint distribution the same as that of the order statistics of $n-1$ random variables $\left(J_{i}\left(Z_{0}^{1 / \alpha}\right):=\left(\left(1-Z_{0}\right) U_{i}+Z_{0}\right)^{-1 / \alpha}\right)_{1 \leq i \leq n-1}$, where the $U_{i}$ s are i.i.d. uniform on $[0,1]$. From this, we retrieve (4.7) by replacing $Z_{0}$ with $u^{\alpha}$.
(a) (ii) The proof of (4.8) is easier because the analogous conditional expectation to (4.9) is of the form

$$
E\left(f_{t}\left(X_{0}, Z_{0}, \ldots, Z_{n-1}\right) \mid \Gamma_{r+n}\right)
$$

which converges as $t \downarrow 0$ to

$$
E\left(f\left(Z_{0}^{-1 / \alpha}, \ldots, Z_{n-1}^{-1 / \alpha}\right) \mid \Gamma_{r+n}\right)=E\left(f\left(Z_{0}^{-1 / \alpha}, \ldots, Z_{n-1}^{-1 / \alpha}\right)\right)
$$

after applying dominated convergence and Lemma 2.1.
Part (b) follows from similar arguments as in Part (ii) of Proposition 4.1.
Remark 4.2. (i) In Part (a)(i) of Proposition 4.2 the $x_{0}$ variable is superfluous, but it is relevant in Part (a)(ii).
(ii) If we make the convention that $B_{0, n} \equiv 0$ a.s., set $u=0$ in (3.8), and identify $\left(J_{i}(0)\right)$ with a sequence $\left(L_{i}\right)$ of i.i.d. random variables, each having the distribution defined in (3.10), we get
the case $r=0$ of (4.8); namely, for $x_{k} \geq 1,0 \leq k \leq n-1, n=2,3, \ldots$, and $x>0$,

$$
\begin{equation*}
\lim _{t \downarrow 0} \mathrm{P}\left(\frac{\Delta_{t}^{(k)}}{\Delta_{t}^{(n)}}>x_{k}, 1 \leq k \leq n-1 \mid \Delta_{t}^{(n)}=x\right)=\mathrm{P}\left(L_{n-1}^{(k)}>x_{k}, 1 \leq k \leq n-1\right), \tag{4.10}
\end{equation*}
$$

where $L_{n-1}^{(1)} \geq L_{n-1}^{(2)} \ldots \geq L_{n-1}^{(n-1)}$ are the decreasing order statistics of $\left(L_{i}\right)_{1 \leq i \leq n-1}$, when $\left(L_{i}\right)_{i \geq 1}$ are i.i.d. random variables each having the distribution in (3.10). Eq. (4.10) can of course be proved directly.
(iii) The case $r \in \mathbb{N}, n=1$, in $\operatorname{Part}(\mathrm{a})$ (i) of Proposition 4.2, is covered by setting $n=r+1$, and $x_{1}=\cdots=x_{r-1}=1$ when $r>1$, in (4.10), to get

$$
\begin{equation*}
\lim _{t \downarrow 0} \mathrm{P}\left(\left.\frac{\Delta_{t}^{(r)}}{\Delta_{t}^{(r+1)}}>x_{r} \right\rvert\, \Delta_{t}^{(r+1)}=x\right)=\mathrm{P}\left(L_{r}^{(r)}>x_{r}\right)=x_{r}^{-r \alpha} \tag{4.11}
\end{equation*}
$$

for $x_{r} \geq 1$ and $x>0$. Here $L_{r}^{(r)} \stackrel{\mathrm{D}}{=} \min _{1 \leq i \leq r} L_{i}$, where $\left(L_{i}\right)_{i \geq 1}$ are as in Part (ii). Note that $L_{r}^{(r)} \stackrel{\mathrm{D}}{=} \mathrm{B}_{r, 1}^{-1 / \alpha}$.
(iv) Convergence of the conditional distributions in (4.8), (4.10), and (4.11), together with

$$
\begin{equation*}
\lim _{t \downarrow 0} \mathrm{P}\left(t \bar{\Pi}\left(\Delta_{t}^{(r+j)}\right) \leq x_{r+j}, 0 \leq j \leq n\right)=\mathrm{P}\left(\Gamma_{r+j} \leq x_{r+j}, 0 \leq j \leq n\right) \tag{4.12}
\end{equation*}
$$

for $0 \leq x_{r} \leq \cdots \leq x_{r+n}$, implies convergence of the corresponding joint, and hence marginal, distributions. Since the right hand sides of (4.8), (4.10), and (4.11) do not depend on $x$, independence obtains in the corresponding limiting joint distributions. To verify (4.12), observe that we have, for $j=0,1, \ldots, n$,

$$
t \bar{\Pi}\left(\Delta_{t}^{(r+j)}\right) \stackrel{\mathrm{D}}{=} t \bar{\Pi}\left(\bar{\Pi} \leftarrow\left(\Gamma_{r+j} / t\right)\right) \rightarrow \Gamma_{r+j}
$$

by (2.7), where the convergence is almost sure as $t \downarrow 0$.
(v) It is essential in Proposition 4.2 to assume $\bar{\Pi}$ is continuous. When $\bar{\Pi}(x)$ has a jump at $x$, then $\bar{\Pi} \leftarrow$ is constant on the interval $[\bar{\Pi}(x), \bar{\Pi}(x-))$, and $\bar{\Pi}$ and $\bar{\Pi} \leftarrow$ are not in 1-1 correspondence. While it is true in general that $\sigma\left(\Delta_{t}^{(i)}\right) \subset \sigma\left(\Gamma_{i}\right)$, the reverse inclusion is not true. In fact it is possible to construct a counterexample (which we omit) to show that the convergence in (4.7) does not hold when $\bar{\Pi}$ has discontinuities.

In the next proposition we need to assume even more in Part (ii); that $\bar{\Pi}$ has a continuous derivative.

Proposition 4.3 (Ratios Smaller than 1). Suppose $\bar{\Pi}(\cdot) \in R V_{0}(-\alpha)$ with $0<\alpha<\infty$ and is continuous. Take $r, n \in \mathbb{N}$.
(i) For each $x>0$ and $w \in(0,1)$

$$
\begin{equation*}
\lim _{t \downarrow 0} \mathrm{P}\left(W_{r, n}(t) \leq w \mid \Delta_{t}^{(r+n)}=x\right)=\mathrm{P}\left(B_{r, n}^{1 / \alpha} \leq w\right) \tag{4.13}
\end{equation*}
$$

(ii) Assume in addition that $\bar{\Pi}(x)$ has a continuous derivative $p(x)$ at each $x>0$ which satisfies

$$
\begin{equation*}
\lim _{x \downarrow 0} \frac{x p(x)}{\bar{\Pi}(x)}=\lim _{x \downarrow 0} \frac{x\left|\bar{\Pi}^{\prime}(x)\right|}{\bar{\Pi}(x)}=\alpha \tag{4.14}
\end{equation*}
$$

Then for each $u>0$ and $w \in(0,1)$

$$
\begin{equation*}
\lim _{t \downarrow 0} \mathrm{P}\left(\Delta_{t}^{(r)}>u \bar{\Pi}^{\leftarrow}(1 / t) \mid W_{r, n}(t)=w\right)=\mathrm{P}\left(\Gamma_{r+n} \leq(u w)^{-\alpha}\right) . \tag{4.15}
\end{equation*}
$$

Remark 4.3. (i) Taking expectations in (4.13) gives, as $t \downarrow 0$,

$$
\begin{equation*}
W_{r, n}(t)=\frac{\Delta_{t}^{(r+n)}}{\Delta_{t}^{(r)}} \xrightarrow{\mathrm{D}} B_{r, n}^{1 / \alpha} . \tag{4.16}
\end{equation*}
$$

From (4.10) with $n$ replaced by $r+n$, we thus have the various alternatives

$$
W_{r, n}(t) \xrightarrow{\mathrm{D}} W_{r, n} \stackrel{\mathrm{D}}{=} B_{r, n}^{1 / \alpha} \stackrel{\mathrm{D}}{=} K_{r+n-1}^{(n)} \stackrel{\mathrm{D}}{=} 1 / L_{r+n-1}^{(r)}, \text { as } t \downarrow 0,
$$

where $K_{r+n-1}^{(n)}$ is the $n$th largest of i.i.d. random variables $\left(K_{i}\right)_{1 \leq i \leq r+n-1}$, each with distribution $\mathrm{P}\left(K_{1} \leq w\right)=w^{\alpha}, w \in(0,1)$, and $L_{r+n-1}^{(r)}$ is the $r$ th largest of i.i.d. random variables $\left(L_{i}\right)_{1 \leq i \leq r+n-1}$, each having the distribution in (3.10). Note that $L \stackrel{\mathrm{D}}{=} 1 / K$.
(ii) Again continuity of $\bar{\Pi}$ is essential for (4.13). (4.14) can be thought of as a sufficient condition for a kind of von Mises condition at 0, cf. [24], p. 63.

Proof of Proposition 4.3. Assume throughout that $\bar{\Pi}(\cdot) \in R V_{0}(-\alpha)$ with $0<\alpha<\infty$ and is continuous.
(i) For $g: \mathbb{R}_{+} \mapsto \mathbb{R}_{+}$, bounded and continuous, we have, as $t \downarrow 0$,

$$
\begin{aligned}
\mathrm{E}\left(g\left(W_{r, n}(t)\right) \mid \Delta_{t}^{(r+n)}\right) & =\mathrm{E}\left(\left.g\left(\frac{\bar{\Pi} t^{\leftarrow}\left(t^{-1} \Gamma_{r+n}\right)}{\bar{\Pi} \leftarrow\left(t^{-1} \Gamma_{r+n}\left(\Gamma_{r} / \Gamma_{r+n}\right)\right)}\right) \right\rvert\, \Gamma_{r+n}\right) \\
& \rightarrow \mathrm{E}\left(\left.g\left(\left(\frac{\Gamma_{r+n}}{\Gamma_{r}}\right)^{-1 / \alpha}\right) \right\rvert\, \Gamma_{r+n}\right)=\mathrm{E}\left(g\left(\left(\frac{\Gamma_{r+n}}{\Gamma_{r}}\right)^{-1 / \alpha}\right)\right),
\end{aligned}
$$

where the last equation follows from Lemma 2.1. This gives (4.13).
(ii) Assume in addition that $\bar{\Pi}(x)$ has a continuous derivative $p(\cdot)$ satisfying (4.14). Take $u>0$ and $0<w<1$ and hold $t>0$ fixed at first. Write

$$
\begin{align*}
& \mathrm{P}\left(\Delta_{t}^{(r)}>u \bar{\Pi} \leftarrow(1 / t) \mid W_{r, n}(t)=w\right)= \\
& \mathrm{P}\left(\bar{\Pi} \leftarrow\left(\Gamma_{r} / t\right)>u \bar{\Pi} \leftarrow(1 / t) \left\lvert\, \frac{\bar{\Pi} \leftarrow\left(\Gamma_{r+n} / t\right)}{\bar{\Pi} \leftarrow\left(\Gamma_{r} / t\right)}=w\right.\right)= \\
& \lim _{\varepsilon \downarrow 0} \frac{\mathrm{P}\left(\Gamma_{r}<t \bar{\Pi}(u \bar{\Pi} \leftarrow(1 / t)),(w-\varepsilon) \bar{\Pi} \leftarrow\left(\Gamma_{r} / t\right)<\bar{\Pi} \leftarrow\left(\Gamma_{r+n} / t\right)<(w+\varepsilon) \bar{\Pi} \leftarrow\left(\Gamma_{r} / t\right)\right)}{\mathrm{P}\left((w-\varepsilon) \bar{\Pi} \leftarrow\left(\Gamma_{r} / t\right)<\bar{\Pi}^{\leftarrow}\left(\Gamma_{r+n} / t\right)<(w+\varepsilon) \bar{\Pi}^{\leftarrow}\left(\Gamma_{r} / t\right)\right)} . \tag{4.17}
\end{align*}
$$

We can represent $\Gamma_{r+n}=\Gamma_{r}+\widetilde{\Gamma}_{n}$, where $\widetilde{\Gamma}_{n}$ is a $\operatorname{Gamma}(n, 1)$ random variable independent of $\Gamma_{r}$, so the numerator in (4.17) can be written as

$$
\begin{aligned}
& \mathrm{P}\left(\Gamma_{r}<t \bar{\Pi}\left(u \bar{\Pi} \bar{\Pi}^{\leftarrow}(1 / t)\right), t \bar{\Pi}\left((w+\varepsilon) \bar{\Pi} \leftarrow\left(\Gamma_{r} / t\right)\right)<\Gamma_{r+n}<t \bar{\Pi}\left((w-\varepsilon) \bar{\Pi} \leftarrow\left(\Gamma_{r} / t\right)\right)\right) \\
&=\int_{y>0} \mathbf{1}\{y<t \bar{\Pi}(u \bar{\Pi} \leftarrow(1 / t))\} \\
& \times \mathrm{P}(t \bar{\Pi}((w+\varepsilon) \bar{\Pi} \leftarrow(y / t))<y \\
&\left.+\widetilde{\Gamma}_{n}<t \bar{\Pi}((w-\varepsilon) \bar{\Pi} \leftarrow(y / t))\right) \mathrm{P}\left(\Gamma_{r} \in \mathrm{~d} y\right),
\end{aligned}
$$

and similarly for the denominator (omitting the indicator function). For brevity let

$$
a_{t}(\varepsilon, y):=t \bar{\Pi}((w+\varepsilon) \bar{\Pi} \leftarrow(y / t))<b_{t}(\varepsilon, y):=t \bar{\Pi}((w-\varepsilon) \bar{\Pi} \leftarrow(y / t)),
$$

and let $f_{n}(z)$ denote the bounded, continuous, density of $\Gamma_{n}$. Then for the limit in (4.17) we need the limit as $\varepsilon \downarrow 0$ of the ratio

$$
\begin{equation*}
\frac{\int_{y>0} \mathbf{1}\left\{y<t \bar{\Pi}\left(u \bar{\Pi}^{\leftarrow}(1 / t)\right)\right\} \int_{a_{t}(\varepsilon, y)-y}^{b_{t}(\varepsilon, y)-y} f_{n}(z) \mathrm{d} z \mathrm{P}\left(\Gamma_{r} \in \mathrm{~d} y\right)}{\int_{y>0} \int_{a_{t}(\varepsilon, y)-y}^{b_{t}(, y)-y} f_{n}(z) \mathrm{d} z \mathrm{P}\left(\Gamma_{r} \in \mathrm{~d} y\right)} . \tag{4.18}
\end{equation*}
$$

With the differentiability assumption on $\bar{\Pi}$, both numerator and denominator here tend to 0 as $\varepsilon \downarrow 0$, because both $a_{t}(\varepsilon, y)$ and $b_{t}(\varepsilon, y)$ tend to $t \bar{\Pi}(w \bar{\Pi} \leftarrow(y / t))$, so we use L'Hospital's rule to evaluate the limit of the ratio. This limit has numerator

$$
\begin{align*}
\int_{y>0} \mathbf{1}\left\{y<t \bar{\Pi}\left(u \bar{\Pi}^{\leftarrow}(1 / t)\right)\right\} & 2 f_{n}(t \bar{\Pi}(w \bar{\Pi} \leftarrow(y / t))-y) \\
& \times t p(w \bar{\Pi} \leftarrow(y / t)) \bar{\Pi} \leftarrow(y / t) \mathrm{P}\left(\Gamma_{r} \in \mathrm{~d} y\right), \tag{4.19}
\end{align*}
$$

and similarly for the denominator (omitting the indicator function). The resulting ratio is an evaluation of the conditional probability we started with in (4.17).

Now let $t \downarrow 0$ in (4.19). The continuity of $f_{n}$ and (2.7) give

$$
\lim _{t \downarrow 0} f_{n}(t \bar{\Pi}(w \bar{\Pi} \leftarrow(y / t))-y)=f_{n}\left(w^{-\alpha} y-y\right), 0<w<1, y>0
$$

and substituting $x=w \bar{\Pi} \leftarrow(y / t)$ in (4.14) and use of (2.7) again give

$$
\lim _{t \downarrow 0} t p(w \bar{\Pi} \leftarrow(y / t)) \bar{\Pi} \leftarrow(y / t)=\alpha w^{-\alpha-1} y, w>0, y>0 .
$$

So via (4.19) we get for the numerator of the limit of the conditional probability in (4.17) the integral

$$
\begin{align*}
& \int_{y>0} \mathbf{1}\left\{y<u^{-\alpha}\right\} f_{n}\left(\left(w^{-\alpha}-1\right) y\right) \cdot \alpha w^{-\alpha-1} y \cdot\left(\frac{e^{-y} y^{r-1} \mathrm{~d} y}{\Gamma(r)}\right) \\
& =\frac{1}{\Gamma(n) \Gamma(r)} \int_{0}^{u^{-\alpha}} e^{-\left(w^{-\alpha}-1\right) y}\left(\left(w^{-\alpha}-1\right) y\right)^{n-1} \cdot \alpha w^{-\alpha-1} y \cdot e^{-y} y^{r-1} \mathrm{~d} y \tag{4.20}
\end{align*}
$$

and similarly for the denominator, but with $u \equiv 0$. The denominator is thus

$$
\frac{\left(w^{-\alpha}-1\right)^{n-1} \alpha w^{-\alpha-1}}{\Gamma(n) \Gamma(r)} \int_{0}^{\infty} e^{-w^{-\alpha} y} y^{n+r-1} \mathrm{~d} y=\frac{\alpha\left(w^{-\alpha}-1\right)^{n-1} w^{(n+r-1) \alpha-1} \Gamma(n+r)}{\Gamma(n) \Gamma(r)}
$$

which is exactly equal to $f_{W_{r, n}}(w)$ in (4.5).
Now we evaluate the conditional probability $\mathrm{P}\left(\Gamma_{r} \leq x \mid W_{r, n}=w\right)$. Use the fact that $W_{r, n} \stackrel{\mathrm{D}}{=}\left(\left(\Gamma_{r}+\widetilde{\Gamma}_{n}\right) / \Gamma_{r}\right)^{-1 / \alpha}$ and, for any $x>0$,

$$
\mathrm{P}\left(\Gamma_{r} \leq x, \Gamma_{r}+\widetilde{\Gamma}_{n} \geq w^{-\alpha} \Gamma_{r}\right)=\int_{0 \leq y \leq x} \int_{x \geq w^{-\alpha} y-y}\left(\frac{e^{-x} x^{n-1}}{\Gamma(n)}\right) \mathrm{d} x\left(\frac{e^{-y} y^{r-1}}{\Gamma(r)}\right) \mathrm{d} y
$$

Differentiate with respect to $w$ and divide by the density of $W_{r, n}$ to get the conditional probability $\mathrm{P}\left(\Gamma_{r} \leq x \mid W_{r, n}=w\right)$ equal to

$$
\begin{equation*}
\frac{1}{f_{W_{r, n}}(w)} \int_{y=0}^{x}\left(\frac{e^{-\left(w^{-\alpha}-1\right) y}\left(w^{-\alpha}-1\right)^{n-1} \alpha w^{-\alpha-1} y^{n}}{\Gamma(n)}\right)\left(\frac{e^{-y} y^{r-1}}{\Gamma(r)}\right) \mathrm{d} y . \tag{4.21}
\end{equation*}
$$

With $x \equiv u^{-\alpha}$ this is exactly the expression in (4.20), divided by $f_{W_{r, n}}(w)$, which, as we showed, is the limit of the conditional probability in (4.17) (and in (4.15)).

Finally, substituting for $f_{W_{r, n}}(w)$ from (4.5), we can simplify (4.21) (with $x$ replaced by $u^{-\alpha}$ ) as

$$
\int_{0}^{u^{-\alpha}} w^{-\alpha(r+n)} \frac{e^{-w^{-\alpha} y} y^{r+n-1}}{\Gamma(r+n)} \mathrm{d} y=\int_{0}^{(u w)^{-\alpha}} e^{-y} \frac{y^{r+n-1}}{\Gamma(r+n)} \mathrm{d} y,
$$

which is the right hand side of (4.15).
Proposition 4.4 (Ratios Greater than 1). Let $\left\{\Gamma_{j}^{-1 / \alpha}, j \geq 1\right\}$ be the ordered points of a Poisson point process with intensity measure $\Lambda(\mathrm{d} x)=\alpha x^{-\alpha-1} \mathrm{~d} x \mathbf{1}_{\{x>0\}}, \alpha>0$. Let $r, n \in \mathbb{N}, 0<u<1$, $\lambda>0$. Then we have the conditional Laplace transform

$$
\begin{equation*}
\mathrm{E}\left(\left.\exp \left(-\lambda \sum_{i=1}^{n-1}\left(\frac{\Gamma_{r+i}}{\Gamma_{r+n}}\right)^{-1 / \alpha}\right) \right\rvert\,\left(\frac{\Gamma_{r+n}}{\Gamma_{r}}\right)^{-1 / \alpha}=u\right)=(\Phi(\lambda, u))^{n-1} \tag{4.22}
\end{equation*}
$$

where

$$
\Phi(\lambda, u)=\frac{\int_{1}^{1 / u} e^{-\lambda x} \Lambda(\mathrm{~d} x)}{1-u^{\alpha}}
$$

When $r=0$, the sum of ratios of jumps greater than 1 has representation

$$
\begin{equation*}
\sum_{i=1}^{n-1}\left(\frac{\Gamma_{i}}{\Gamma_{n}}\right)^{-1 / \alpha} \stackrel{\mathrm{D}}{=} \sum_{i=1}^{n-1} L_{i} \tag{4.23}
\end{equation*}
$$

where the $L_{i}$ are i.i.d random variables each with the same distribution as $\Gamma_{1} / \Gamma_{2}$; namely, $\mathrm{P}\left(L_{1} \in \mathrm{~d} x\right)=\Lambda(\mathrm{d} x) \mathbf{1}_{\{x>1\}}$.

Remark 4.4. The representation (4.23) of the sum of ratios of the ordered jumps of a stable subordinator as a random walk in $n$ provides the impetus for further work in large trimming results in the spirit of the investigations in [5].

Proof of Proposition 4.4. Equality (4.22) can be read from the order statistics property of the homogeneous Poisson process when $r>0$ and from (3.9) when $r=0$.

## 5. Converse results

Theorem 5.1 gives converses to the previous results.
Theorem 5.1 (Converse Results: Ratios Bigger than 1). Suppose, for some $r \in \mathbb{N}, n \in \mathbb{N}$, $\Delta_{t}^{(r)} / \Delta_{t}^{(r+n)} \xrightarrow{\mathrm{D}} Y$, as $t \downarrow 0$, for an extended value random variable ${ }^{1} \quad Y \geq 1$. Then one of the following holds:
(i) $\mathrm{P}(1<Y<\infty)>0$, in which case $\bar{\Pi}(\cdot) \in R V_{0}(-\alpha)$ with $0<\alpha<\infty$;
(ii) $\mathrm{P}(Y=1)=1$, in which case $\bar{\Pi}$ is rapidly varying at 0 ;
(iii) $\mathrm{P}(Y=\infty)=1$, in which case $\bar{\Pi}$ is slowly varying at 0 .

Remark 5.1 (Converse Results: Ratios Smaller than 1). Analogous results to Theorem 5.1 for ratios smaller than 1 follow by taking reciprocals. Write $\Delta_{t}^{(r+n)} / \Delta_{t}^{(r)}=\left(\Delta_{t}^{(r)} / \Delta_{t}^{(r+n)}\right)^{-1}$ and apply the theorem, replacing $Y$ by $Y^{-1}$, and making the obvious interpretations in Parts (i), (ii) and (iii) of the theorem.

[^2]Proof of Theorem 5.1. Assume for some $r \in \mathbb{N}, n \in \mathbb{N}$,

$$
\begin{equation*}
\frac{\Delta_{t}^{(r)}}{\Delta_{t}^{(r+n)}} \xrightarrow{\mathrm{D}} Y, \text { as } t \downarrow 0, \tag{5.1}
\end{equation*}
$$

where $Y$ is an extended random variable with distribution $G$, say, on $[1, \infty]$. The proof that follows is similar in style to that of Kevei and Mason [13] who considered ratios of successive jumps, that is, the case $n=1$. When $n>1$, some rather different arguments are needed at some places.

Keep $u$ fixed in $(0,1)$ throughout the remainder of the proof and use (2.3) to write

$$
\begin{aligned}
\mathrm{P}\left(\Delta_{t}^{(r+n)}<u \Delta_{t}^{(r)}\right) & =\mathrm{P}\left(\bar{\Pi} \leftarrow\left(\left(\Gamma_{r}+\widetilde{\Gamma}_{n}\right) / t\right)<u \bar{\Pi} \leftarrow\left(\Gamma_{r} / t\right)\right) \\
& =\int_{y \geq 0} \mathrm{P}\left(\widetilde{\Gamma}_{n}>t \bar{\Pi}(u \bar{\Pi} \leftarrow(y / t))-y\right) \mathrm{P}\left(\Gamma_{r} \in \mathrm{~d} y\right),
\end{aligned}
$$

where $\Gamma_{r}$ and $\widetilde{\Gamma}_{n}$ are independent Gamma random variables. Substituting for their densities gives

$$
\begin{align*}
& \mathrm{P}\left(\Delta_{t}^{(r+n)}<u \Delta_{t}^{(r)}\right)=\int_{y>0} \int_{z>t \bar{\Pi}(u \bar{\Pi} \leftarrow(y / t))-y}\left(\frac{e^{-z} z^{n-1}}{\Gamma(n)}\right) \mathrm{d} z\left(\frac{e^{-y} y^{r-1}}{\Gamma(r)}\right) \mathrm{d} y \\
& =\frac{t^{r+n}}{\Gamma(r+n)} \int_{y>0} \int_{z>\bar{\Pi}(u \bar{\Pi} \leftarrow(y))}\left(\frac{e^{-t z}(z-y)^{n-1}}{B(r, n)}\right) \mathrm{d} z y^{r-1} \mathrm{~d} y \\
& =\frac{t^{r+n}}{\Gamma(r+n)} \int_{z>0} \int_{0<y<\bar{\Pi}\left(\bar{\Pi}^{\leftarrow}(z) / u\right) / z}\left(\frac{(1-y)^{n-1} y^{r-1}}{B(r, n)}\right) \mathrm{d} y e^{-t z} z^{r+n-1} \mathrm{~d} z . \tag{5.2}
\end{align*}
$$

Here note that, since $u<1$, we have $t \bar{\Pi}(u \bar{\Pi} \leftarrow(y / t)) \geq t \bar{\Pi}(\bar{\Pi} \leftarrow(y / t)-) \geq y$, and $\bar{\Pi}(\bar{\Pi} \leftarrow(z) / u) / z \leq \bar{\Pi}(\bar{\Pi} \leftarrow(z)) / z \leq 1$. We recognise the inner integral ${ }^{2}$ in (5.2) as the incomplete Beta function $B(r, n ; \bar{\Pi}(\bar{\Pi} \leftarrow(z) / u) / z)$ (see (2.5)). By assumption (5.1), the expression in (5.2) tends to $\bar{G}(1 / u):=\mathrm{P}(Y>1 / u)$ as $t \downarrow 0$, at continuity points of $\bar{G}$. To simplify the notation, from this point on let $x=1 / u>1$. Let

$$
U_{x}(z):=\int_{0}^{z} B\left(r, n ; \bar{\Pi}\left(\bar{\Pi}^{\leftarrow}(v) x\right) / v\right) v^{r+n-1} \mathrm{~d} v, z>0 .
$$

Then from (5.2),

$$
t^{r+n} \int_{z>0} e^{-t z} U_{x}(\mathrm{~d} z) \rightarrow \Gamma(r+n) \bar{G}(x), \text { as } t \downarrow 0
$$

at continuity points of $\bar{G}$, which by Thm. 1.7.1 p. 37 of Bingham et al. [3] implies

$$
z^{-r-n} U_{x}(z) \rightarrow \Gamma(r+n) \bar{G}(x) / \Gamma(r+n+1)=\bar{G}(x) /(r+n), \text { as } z \rightarrow \infty .
$$

Write this as

$$
\begin{equation*}
\frac{1}{z^{r+n}} \int_{0}^{z} b_{x}(v) v^{r+n-1} \mathrm{~d} v \rightarrow \frac{\bar{G}(x)}{r+n}, \text { as } z \rightarrow \infty \tag{5.3}
\end{equation*}
$$

where

$$
b_{x}(v)=B\left(r, n ; f_{x}(v)\right)=\frac{1}{B(r, n)} \int_{0}^{f_{x}(v)} y^{r-1}(1-y)^{n-1} \mathrm{~d} y=: \int_{0}^{f_{x}(v)} p(y) \mathrm{d} y
$$

[^3]with $f_{x}(v):=\bar{\Pi}\left(\bar{\Pi}{ }^{\leftarrow}(v) x\right) / v, v>0$ and $p(y):=y^{r-1}(1-y)^{n-1} / B(r, n), 0 \leq y \leq 1$. Note that $x$ is kept fixed in $b_{x}(v)$ and $f_{x}(v)$. We have $0 \leq f_{x}(v) \leq 1$, so $0 \leq b_{x}(v) \leq 1$, for all $v>0$. (5.3) implies
\[

$$
\begin{equation*}
\frac{1}{z^{r+n}} \int_{z}^{\lambda z} b_{x}(v) v^{r+n-1} \mathrm{~d} v=\int_{1}^{\lambda} b_{x}(v z) v^{r+n-1} \mathrm{~d} v \rightarrow \frac{\left(\lambda^{r+n}-1\right) \bar{G}(x)}{r+n}, \text { as } z \rightarrow \infty \tag{5.4}
\end{equation*}
$$

\]

for any $\lambda>1$ and each fixed $x>0$.
Functions $b_{x}(v), f_{x}(v)$, are not necessarily monotone but are of bounded variation (BV) on finite intervals bounded away from 0 . To see this, observe that the function $m_{x}(v):=v f_{x}(v)=$ $\bar{\Pi}(\bar{\Pi} \leftarrow(v) x)$ is nondecreasing in $v$ and

$$
\left|\mathrm{d} f_{x}(v)\right|=\left|\frac{\mathrm{d} m_{x}(v)}{v}-\frac{m_{x}(v) \mathrm{d} v}{v^{2}}\right| \leq \frac{\mathrm{d} m_{x}(v)}{v}+\frac{\mathrm{d} v}{v}
$$

thus, with $p_{0}:=\sup _{0 \leq y \leq 1} p(y)$,

$$
\left|\mathrm{d} b_{x}(v)\right|=\left|p\left(f_{x}(v)\right) \mathrm{d} f_{x}(v)\right| \leq p_{0}\left(\frac{\mathrm{~d} m_{x}(v)}{v}+\frac{\mathrm{d} v}{v}\right)
$$

and the RHS is integrable over $v \in[\delta, z]$, for any $0<\delta<z$. So $f_{x}$ and $b_{x}$ are of bounded variation on $[\delta, z]$ for any $0<\delta<z$. Take any sequence $z_{k} \rightarrow \infty$. By Helly's theorem for finite measures we can find a subsequence, also denoted $z_{k}$, possibly depending on $x$, such that

$$
b_{x}\left(v z_{k}\right) \rightarrow g_{x}(v), v>0, \text { as } k \rightarrow \infty,
$$

at continuity points of $g$, for a function $g_{x}(v) \in[0,1]$. Using dominated convergence in (5.4) we get

$$
\int_{1}^{\lambda} g_{x}(v) v^{r+n-1} \mathrm{~d} v=\frac{\left(\lambda^{r+n}-1\right) \bar{G}(x)}{r+n}=\bar{G}(x) \int_{1}^{\lambda} v^{r+n-1} \mathrm{~d} v .
$$

This holds for all $\lambda>1$ and so implies $g_{x}(v)=\bar{G}(x)$, for all $v>1, x>0$, not depending on the choice of subsequence. Thus we deduce that

$$
b_{x}(v z)=\int_{0}^{f_{x}(v z)} p(y) \mathrm{d} y \rightarrow \bar{G}(x)
$$

as $z \rightarrow \infty$, at continuity points of $\bar{G}$, for all $v>1$. Take $v=2$. Now $f_{x}(2 z)$ is monotone in $x$ for each $z$, so by Helly's theorem again each sequence $z_{k} \rightarrow \infty$ contains a further subsequence, also denoted $z_{k}$, such that $f_{x}\left(2 z_{k}\right) \rightarrow h(x) \in[0,1]$, as $k \rightarrow \infty$, at continuity points of $h(x)$. Thus we obtain

$$
\begin{equation*}
\int_{0}^{h(x)} p(y) \mathrm{d} y=\bar{G}(x) \tag{5.5}
\end{equation*}
$$

at continuity points of $h$. Again the limit does not depend on the choice of subsequence. This identifies $h(x)$ as $I \leftarrow(\bar{G}(x))$, where $I^{\leftarrow}(\cdot)$ is the unique inverse function to the continuous strictly increasing function $I(\cdot)=\int_{0}^{\cdot} p(y) \mathrm{d} y$. Thus, continuity points of $h$ are points of increase of $G$. Define

$$
\mathcal{A}:=\{x \geq 1: x \text { is a continuity point and a point of increase of } G\} .
$$

We conclude that

$$
\begin{equation*}
\lim _{z \rightarrow \infty} \frac{\bar{\Pi}\left(\bar{\Pi}^{\leftarrow}(z) x\right)}{z}=\lim _{z \rightarrow \infty} f_{x}(z)=\lim _{z \rightarrow \infty} f_{x}(2 z)=h(x), \text { for all } x \in \mathcal{A} \tag{5.6}
\end{equation*}
$$

where $h$ satisfies (5.5). (5.6) is exactly analogous to Eq. (2.10) of Kevei and Mason [13] and we can follow their arguments henceforth to finish the converse part of the proof. There are three alternatives.
(i) $\mathrm{P}(1<Y<\infty)>0$. In this case $\bar{G}$ has at least one point of decrease in $(1, \infty)$, say $x$, and a neighbourhood $(x-\varepsilon, x+\varepsilon)$ for some $\varepsilon>0$, such that $\bar{G}(y)>0$ for all $y$ in the neighbourhood. Kevei and Mason [13] gave a careful analysis of this situation, showing that it leads to $\bar{\Pi}(\cdot) \in R V_{0}(-\alpha)$ with $0<\alpha<\infty$.
(ii) $\mathrm{P}(Y=1)=1$. This means that $\mathrm{P}(Y>x)=\bar{G}(x)=0$ for all $x>1$, so $\int_{0}^{h(x)} p(y) \mathrm{d} y=0$ and

$$
\lim _{z \rightarrow \infty} \frac{\bar{\Pi}\left(\bar{\Pi}^{\leftarrow}(z) x\right)}{z} \leq \lim _{z \rightarrow \infty} \frac{\bar{\Pi}(\bar{\Pi} \leftarrow(z) x)}{\bar{\Pi}\left(\bar{\Pi}^{\leftarrow}(z)\right)}=0
$$

for all $x>1$. Thus $\bar{\Pi}$ is rapidly varying at 0 .
(iii) $Y=\infty$ a.s. This means that $\mathrm{P}(Y>x)=\bar{G}(x)=1$ for all $x>1$, so $\int_{0}^{h(x)} p(y) \mathrm{d} y=1$ and

$$
\lim _{z \rightarrow \infty} \frac{\bar{\Pi}(\bar{\Pi} \leftarrow(z) x)}{z}=1
$$

for all $x>1$. This leads to $\bar{\Pi}$ slowly varying at 0 as shown in [13], and completes the proof.

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[^2]:    ${ }^{1}$ A random variable that may take the value $+\infty$ with positive probability.

[^3]:    ${ }^{2}$ When $n=1$ the inner integral can be evaluated explicitly and the analysis takes a simpler form, as in [13].

