# DEGREE GROWTH RATES AND INDEX ESTIMATION IN A DIRECTED PREFERENTIAL ATTACHMENT MODEL 

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#### Abstract

Preferential attachment is widely used to model power-law behavior of degree distributions in both directed and undirected networks. In a directed preferential attachment model, despite the well-known marginal power-law degree distributions, not much investigation has been done on the joint behavior of the in- and out-degree growth. Also, statistical estimates of the marginal tail exponent of the power-law degree distribution often use the Hill estimator as one of the key summary statistics, even though no theoretical justification has been given. This paper focuses on convergence of the joint empirical measure for in- and out-degrees and proves the consistency of the Hill estimator. To do this, we first derive the asymptotic behavior of the joint degree sequences by embedding the in- and out-degrees of a fixed node into a pair of switched birth processes with immigration and then establish the convergence of the joint tail empirical measure. From these steps, the consistency of the Hill estimators is obtained.


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## 1. Introduction.

The preferential attachment model generates a growing sequence of random graphs based on the assumption that popular nodes with large degrees attract more edges. Nodes and edges are added to the graph following probabilistic rules. Such mechanism provides a basis for studying the evolution of social networks, collaborator and citation networks, as well as recommender networks, and is applicable to both directed and undirected graphs. Mathematical formulations of the undirected preferential attachment model are available in [ $2,7,22$ ], and those of the directed model can be found in [3, 13]. This paper only considers the directed model where at each stage, a new node is born and either it points to one of the existing nodes or one of the existing nodes attaches to the new node. Results on the degree growth in the undirected case are investigated in [1, 27].

Empirical studies on social network data often reveal that in- and out-degree distributions marginally follow power laws. Theoretically, this is also true for linear preferential attachment models, which makes preferential attachment appealing in network modeling; see $[3,12,13]$ for references. Also, the empirical joint degree frequency converges to the probability mass function (pmf) of a pair of limit random variables that are jointly regularly varying (cf. [13, 19, 20, 26]). However, questions related to joint degree growth and index estimation still remain unresolved. In this paper, we focus on three main problems:

[^0](1) For a fixed node in a linear preferential attachment graph, what is the joint behavior of in- and out-degree as the graph size grows?
(2) What are the convergence properties of the tail empirical joint measure of in- and out-degrees indexed by node?
(3) When estimating the marginal power-law indices of in- and out-degree, can we use the Hill estimator as a consistent estimator?

What is the justification for interest in Hill estimation of power-law indices for network data? Repositories of large network datasets such as KONECT (http://konect.unikoblenz.de/, [14]) provide summary statistics for all the archived network datasets and among the summary statistics are estimates of degree indices computed with Hill estimators, despite the fact that evidence for Hill estimator consistency is scant for network data [27].

Another justification is robust parameter estimation methods in network models based on extreme value techniques. In [23], we couple the Hill estimation of marginal degree distribution tail indices with a minimum distance threshold selection method introduced in [4] and compare this method with the parametric estimation approaches used in [24]. The Hill estimation is more robust against modeling errors and data corruptions. Therefore, an affirmative answer to the third question helps justify all of these inference methodologies.

In the directed case, consistency of the two marginal Hill estimators results from resolving the first two questions, since in a similar vein to [27], we consider the Hill estimator as a functional of the marginal tail empirical measure. So convergence results of marginal tail empirical measures lead to the consistency of Hill estimators by a mapping argument.

To answer the first question about degree behavior of fixed nodes as graph size grows, we mimic in- and out-degree growth of a fixed node using pairs of switched birth processes with immigration (SBI processes). The SBI processes use Bernoulli switching between pairs of independent birth processes with immigration (BI processes). We embed the directed network growth model into a sequence of paired SBI processes. Whenever a new node is added to the network, a new pair of SBI processes is initiated. Using convergence results for BI processes (cf. [17, Chapter 5.11], [21, 27]), we give the joint limits of the in- and out-degrees of a fixed node as well as the joint maximal degree growth. Proving the convergence of the tail empirical joint measure in the second question requires showing concentration results for degree counts compared with expected degree counts. With embedding techniques, we prove the limit distribution of the empirical joint degree frequencies in a way that is different from the one used in [20], and then justify the concentration results.

Our paper is structured as follows. In the rest of this section, we review background on the tail empirical measure and Hill estimator. Section 2 sets up the linear preferential attachment model and formulates the power-law phenomena in network degree distributions. Section 3 summarizes facts about BI processes and introduces the SBI process, which is the foundation of the embedding technique. We analyze the joint in- and out-degree growth in Section 4 by embedding it into a sequence of paired SBI processes and derive convergence results of the in- and out-degrees for a fixed node. Results on the convergence of the joint empirical measure are given in Section 5 and the consistency of Hill estimators for both inand out-degrees is proved in Section 6. Useful concentration results are collected in Section 7.
1.1. Background. Our approach to the Hill estimator considers it as a functional of the tail empirical measure so we start with necessary background and review standard results (cf. [18, Chapter 3.3.5 and 6.1.4]).
1.1.1. Non-standard regular variation. Let $M_{+}\left([0, \infty]^{2} \backslash\{\mathbf{0}\}\right)$ be the set of Radon measures on $[0, \infty]_{+}^{2} \backslash\{\mathbf{0}\}$. Then a random vector $(X, Y)$ is non-standard regularly varying on $[0, \infty]_{+}^{2} \backslash\{\mathbf{0}\}$ if there exist scaling functions $b_{i}(t) \rightarrow \infty, i=1,2$ such that as $t \rightarrow \infty$,

$$
\begin{equation*}
t \mathbf{P}\left[\left(\int_{b_{1}(t)}^{X}, \frac{Y}{b_{2}(t)}\right) \in \cdot\right]\left(\stackrel{v}{\longrightarrow} \nu(\cdot), \quad \text { in } M_{+}\left([0, \infty]^{2} \backslash\{\mathbf{0}\}\right),\right. \tag{1.1}
\end{equation*}
$$

where $\nu(\cdot) \in M_{+}\left([0, \infty]^{2} \backslash\{\mathbf{0}\}\right)$ is called the limit or tail measure $[19,20]$, and " $\xrightarrow{v}$ " denotes the vague convergence of measures in $M_{+}\left([0, \infty]^{2} \backslash\{\mathbf{0}\}\right)$. The phrasing in (1.1) implies the marginal distributions have regularly varying tails.
1.1.2. Hill Estimator. For $x \in(0, \infty]$, define the measure $\epsilon_{x}(\cdot)$ on Borel subsets $A$ of $(0, \infty]$ by

$$
\epsilon_{x}(A)=\left\{\begin{array}{l}
x \in A, \\
x \neq A,
\end{array} \quad \text { for } A \in \mathcal{E}\right.
$$

Let $M_{+}((0, \infty])$ be the set of non-negative Radon measures on $(0, \infty]$. A point measure $m$ is an element of $M_{+}((0, \infty])$ of the form

$$
\begin{equation*}
m=\sum_{i} \not x_{i} \tag{1.2}
\end{equation*}
$$

For $\left\{X_{n}, n \geq 1\right\}$ iid and non-negative with common regularly varying distribution tail $\bar{F} \in R V_{-\iota}, \iota>0$, there exists a sequence $\{b(n)\}$ satisfying $P\left[X_{1}>b(n)\right] \sim 1 / n$, such that for any $k_{n} \rightarrow \infty, k_{n} / n \rightarrow 0$,

$$
\begin{equation*}
\frac{1}{k_{n}} \sum_{i=1}^{n} \epsilon_{X_{i} / b\left(n / k_{n}\right)} \Rightarrow \nu_{\iota}, \quad \text { in } M_{+}((0, \infty]) \tag{1.3}
\end{equation*}
$$

where the limit measure $\nu_{\iota}$ satisfies $\nu_{\iota}(y, \infty]=y^{-\iota}, y>0$.
Define the Hill estimator $H_{k, n}$ based on $k$ upper order statistics of $\left\{X_{1}, \ldots, X_{n}\right\}$ as [10]

$$
\begin{equation*}
H_{k, n}:=\frac{1}{k} \sum_{i=1}^{k} \log \frac{X_{(i)}}{X_{(k+1)}}, \tag{1.4}
\end{equation*}
$$

where $X_{(1)} \geq X_{(2)} \geq \ldots \geq X_{(n)}$ are order statistics of $\left\{X_{i}: 1 \leq i \leq n\right\}$. In the iid case there are many proofs of consistency [5, 6, 9, 15, 16]: For $k=k_{n} \rightarrow \infty, k_{n} / n \rightarrow 0$, we have

$$
\begin{equation*}
H_{k_{n}, n} \xrightarrow{P} 1 / \iota \quad \text { as } n \rightarrow \infty . \tag{1.5}
\end{equation*}
$$

The treatment in [18, Theorem 4.2] approaches consistency by showing (1.5) follows from (1.3) and we follow this approach for the network context where the iid case is inapplicable.
1.1.3. Node degrees. The next section constructs a directed preferential attachment model, and gives behavior of $\left(D_{v}^{\text {in }}(n), D_{v}^{\text {out }}(n)\right)$, the in- and out-degrees of node $v$ at the $n$th stage of construction. These degrees when scaled by appropriate powers of $n$ (see (4.12)) have limits and Theorem 5.4 shows that the degree sequences $\left(D_{v}^{\text {in }}(n), D_{v}^{\text {out }}(n)\right)_{1}(\leq v \leq n$ have a joint tail empirical measure

$$
\begin{equation*}
\frac{1}{k_{n}} \sum_{v} \notin\left(\phi_{v}^{\text {in }}(n) / b_{1}\left(n / k_{n}\right), D_{v}^{\text {out }}(n) / b_{2}\left(n / k_{n}\right)\right) \tag{1.6}
\end{equation*}
$$

that converges weakly to some limit measure in $M_{+}\left([0, \infty]^{2} \backslash\{\mathbf{0}\}\right)$, where $b_{1}(n), b_{2}(n)$ are appropriate power law scaling functions and $k_{n}$ is some intermediate sequence such that

$$
k_{n} / n \rightarrow 0, \quad k_{n} \rightarrow \infty, \quad \text { as } n \rightarrow \infty
$$

It also follows from (1.6) that for some tail indices $\iota_{\text {in }}, \iota_{\text {out }}$, and intermediate sequence $k_{n}$,

$$
\begin{align*}
& \frac{1}{k_{n}} \sum_{v} \epsilon_{D_{v}^{\text {in }}(n) / b_{1}\left(n / k_{n}\right)} \Rightarrow \nu_{\iota_{\text {in }}}, \quad \text { in } M_{+}((0, \infty])  \tag{1.7}\\
& \frac{1}{k_{n}} \sum_{v} \oint_{D_{v}^{\text {out }}(n) / b_{2}\left(n / k_{n}\right)} \Rightarrow \nu_{\iota_{\text {out }}}, \quad \text { in } M_{+}((0, \infty]) \tag{1.8}
\end{align*}
$$

This leads to consistency of the Hill estimator for $\iota_{\text {in }}$ and $\iota_{\text {out }}$.

## 2. Preferential Attachment Models.

2.1. Model setup. Consider $\{G(n), n \geq 1\}$, a growing sequence of preferential attachment graphs. The graph $G(n)$ consists of $n$ nodes, denoted by $[n]:=\{1,2, \ldots, n\}$, and $n$ directed edges; the set of edges of $G(n)$ consisting of ordered pairs of nodes in $[n]$ is denoted by $E(n)$. The initial graph $G(1)$ consists of one node, labeled node 1, with a self loop. Thus node 1 has in- and out-degrees both equal to 1 . For $n \geq 1$, we obtain a new graph $G(n+1)$ by appending a new node $n+1$ and a new directed edge to the existing graph $G(n)$ according to probabilistic rules described below. For $v \in[n],\left(D_{v}^{\text {in }}(n), D_{v}^{\text {out }}(n)\right)$ are the in- and out-degree of node $v$ in $G(n)$. The direction of the new edge in $G(n+1)$ is determined by flipping a 2 -sided coin, which has probabilities $\alpha \in(0,1)$ and $1-\alpha \equiv \gamma$, such that given $G(n)$ and two positive parameters $\delta_{\text {in }}, \delta_{\text {out }}>0$ (not necessarily equal):

- If the coin comes up heads with probability $\alpha$, direct the new edge from the new node $n+1$ to the existing node $v \in[n]$ with probability depending on the in-degree of $v$ in $G(n)$ :

$$
\begin{equation*}
\mathbf{P}(v \in[n] \text { is chosen })=\frac{D_{v}^{\mathrm{in}}(n)+\delta_{\mathrm{i} n}}{\left(1+\delta_{\mathrm{i} n}\right) n} . \tag{2.1}
\end{equation*}
$$

- If the coin comes up tails with probability $\gamma$, direct the new edge from an existing node $v \in[n]$ to the new node $n+1$, with probability depending on the out-degree of $v$ in $G(n)$ :

$$
\begin{equation*}
\mathbf{P}(v \in[n] \text { is chosen })=\frac{D_{v}^{\text {out }}(n)+\delta_{\text {out }}}{\left(1+\delta_{\text {out }}\right) n} \tag{2.2}
\end{equation*}
$$

We refer the two scenarios as $\alpha$ - and $\gamma$-schemes, respectively.
2.1.1. Model construction. One way to formally construct the model which helps with proofs is by using independent exponential random variables (r.v.'s). Define derived parameters

$$
\begin{equation*}
c_{\mathrm{in}}=\frac{\alpha}{1+\delta_{\mathrm{in}}} \quad \text { and } \quad c_{\mathrm{out}}=\frac{\gamma}{1+\delta_{\mathrm{out}}} \tag{2.3}
\end{equation*}
$$

and for $n \geq 1$, we will recursively define what corresponds to the in- and out-degree sequences as random elements of $\left(\mathbb{N}_{+}^{2}\right)^{\infty}$,

$$
\begin{align*}
& \mathcal{D}(n):=\left(\left(D_{1}^{\text {in }}(n), D_{1}^{\text {out }}(n)\right), \ldots,\left(D_{n}^{\text {in }}(n), D_{n}^{\text {out }}(n)\right),(0,0), \ldots\right)(  \tag{2.4}\\
& \text { ion }
\end{align*}
$$

$$
\begin{equation*}
\mathcal{D}(1)=((1,1),(0,0), \ldots) \tag{2.5}
\end{equation*}
$$

corresponding to assuming $G(0)$ has a single node with a s申lf loop. For $k \geq 1$, the recursive definition of $\{\boldsymbol{\mathcal { D }}(n)\}$ uses the variables

$$
\begin{align*}
& \boldsymbol{e}_{k}^{\text {in }}:=((0,0), \ldots,(0,0), \underbrace{(1,0)}_{k \text {-th entry }},(0,0), \ldots),  \tag{2.6}\\
& \boldsymbol{e}_{k}^{\text {out }}:=((0,0), \ldots,(0,0), \underbrace{(0,1)}_{k \text {-th entr }}\}(0,0), \ldots),
\end{align*}
$$

and relies on competitions from exponential alarm clocks based on $\left\{E_{k}^{(n)}: k \geq 1, n \geq 1\right\}$, a sequence of iid standard exponential r.v.'s. Assuming $\mathcal{D}(n)$ has been given, $\mathcal{D}(n+1)$ requires $\mathcal{D}(n)$ and the $2 n$ variables $\left\{E_{j}^{(n)}, j=1, \ldots, 2 n\right\}$ which are independent of $\mathcal{D}(n)$ (which can be checked recursively) and we define

$$
\begin{aligned}
& \bar{E}_{k}^{(n)}:=\frac{E_{k}^{(n)}}{\frac{c_{\text {in }}}{c_{\text {in }}+c_{\text {out }}}\left(D_{k}^{\text {in }}(n)+\delta_{\text {in }}\right)}, \quad k=1, \ldots, n, \\
& \bar{E}_{k}^{(n)}:=\frac{E_{k}^{(n)}}{\frac{c_{\text {out }}}{c_{\text {in }}+c_{\text {out }}}\left(D_{k}^{\text {out }}(n)+\delta_{\text {out }}\right)}, \quad k=n+1, \ldots, 2 n .
\end{aligned}
$$

Conditionally on $\mathcal{D}(n)$, use the $\left\{\bar{E}_{k}^{(n)}: k=1, \ldots, 2 n\right\}$ to create a competition between exponentially distributed alarm clocks. For $\delta_{\mathrm{i} n}, \delta_{\mathrm{out}}>0$ and $n \geq 1$, define choice variables

$$
L_{n+1}=\sum_{l=1}^{n} l \mathbf{1}_{\left\{\bar{E}_{l}^{(n)}<\Lambda_{k=1, k \neq l}^{2 n} \bar{E}_{k}^{(n)}, 1 \leq l \leq n\right\}}+\sum_{l=n+1}^{2 n}\left\{\mathbf{1}_{\left\{\bar{E}_{l}^{(n)}<\Lambda_{k=1, k \neq l}^{2 n} \bar{E}_{k}^{(n)}, n+1 \leq l \leq 2 n\right\}} .\right.
$$

So $L_{n+1}$ is the index of the minimum of $\left\{\bar{E}_{k}^{(n)}, 1 \leq k \leq 2 n\right\}$ indicating the winner of the competition. Also, for $n \geq 1$, define the Bernoulli random variable

$$
B_{n+1}:=\mathbf{1}_{\left\{\bigwedge_{k=1}^{n} \bar{E}_{k}^{(n)}>\bigwedge_{k=n+1}^{2 n} \bar{E}_{k}^{(n)}\right\}}=\mathbf{1}_{\left\{L_{n+1}>n\right\}},
$$

and given $\mathcal{D}(n)$, we have

$$
\begin{equation*}
\mathcal{D}(n+1)=\mathcal{D}(n)+\left(1-B_{n+1}\right) \boldsymbol{e}_{L_{n+1}}^{\text {in }}+B_{n+1} \boldsymbol{e}_{L_{n+1}-n}^{\text {out }}+B_{n+1} \boldsymbol{e}_{n+1}^{\text {in }}+\left(1-B_{n+1}\right) \boldsymbol{e}_{n+1}^{\text {out }} . \tag{2.8}
\end{equation*}
$$

This increments the $L_{n+1}$-st pair by $(1,0)$ if $B_{n+1}=0$ and the $\left(L_{n+1}-n\right)$-th pair by $(0,1)$ if $B_{n+1}=1$; the first case corresponds to an increase of in-degree and the second case to an
increase of out-degree. The recursion also assigns to pair $n+1$ either $(1,0)$ or $(0,1)$ depending on the case. This construction expresses $\mathcal{D}(n+1)$ as a function of $\mathcal{D}(n)$ and something independent, namely $\left\{E_{j}^{(n)}, j=1, \ldots, 2 n\right\}$ and therefore the process $\{\mathcal{D}(n), n \geq 1\}$ is an $\left(\mathbb{N}_{+}^{2}\right)^{\infty}$-valued Markov chain. Also, because of the initialization (2.5), a simple induction argument applied to (2.8) gives the sum of the components satisfies

$$
\begin{equation*}
\sum_{j} D_{j}^{\mathrm{in}}(n)=\sum_{j} \phi_{j}^{\text {out }}(n)=n, \quad n \geq 1 \tag{2.9}
\end{equation*}
$$

Then using (2.3), (2.9) and standard calculations with exponential rv's, we have for $v \in[n]$,

$$
\begin{align*}
& \mathbf{P}\left(\mathcal{D}(n+1)=\boldsymbol{\mathcal { D }}(n)+\boldsymbol{e}_{v}^{\text {in }}+\boldsymbol{e}_{n+1}^{\text {out }} \mid \mathcal{D}(n)\right)=\mathbf{P}\left(L_{n+1}=v \boldsymbol{\mathcal { D }}(n)\right)( \\
& =\mathbf{P} \quad \bar{E}_{v}^{(n)}<\bigwedge_{k=1, k \neq v}^{2 n}\left(\bar{E}_{k}^{(n)} \mathcal{D}(n)\right)\left(=\frac{\alpha\left(D_{v}^{\text {in }}(n)+\delta_{\text {in }}\right)}{\left(1+\delta_{\text {in }}\right) n}\right. \tag{2.10}
\end{align*}
$$

and likewise

$$
\begin{align*}
& \mathbf{P}\left(\boldsymbol{D}(n+1)=\boldsymbol{\mathcal { D }}(n)+\boldsymbol{e}_{v}^{\text {out }}+\boldsymbol{e}_{n+1}^{\text {in }} \boldsymbol{\mathcal { D }}(n)\right) \neq \mathbf{P}\left(L_{n+1}=n+v \mid \mathcal{D}(n)\right) \\
& =\mathbf{P} \quad \bar{E}_{n+v}^{(n)}<\bigwedge_{k=1, k \neq v}^{2 n}\left(\bar{E}_{k}^{(n)} \boldsymbol{\mathcal { D }}(n)\right)=\frac{\gamma\left(D_{\alpha}^{\text {out }}(n)+\delta_{\text {out }}\right)}{\left(1+\delta_{\text {out }}\right) n} . \tag{2.11}
\end{align*}
$$

These probabilities agree with the attachment probabilities (2.1), (2.2) in $\alpha$ - and $\gamma$-schemes, respectively.
2.2. Power-law tails. Suppose $G(n)$ is a random graph generated by the dynamics above after $n$ steps. Let $N_{i, j}(n)$ be the number of nodes in $G(n)$ with in-degree $i$ and out-degree $j$, i.e.

$$
\begin{equation*}
N_{i, j}(n):=\sum_{v \in[n]}\left(1\left\{\left(\phi_{v}^{\text {in }}(n), D_{v}^{\text {out }}(n)\right) \neq(i, j)\right\},\right. \tag{2.12}
\end{equation*}
$$

then $N_{i}^{\mathrm{in}}(n):=\sum_{j} N_{i, j}(n)$ and $N_{>i}^{\mathrm{in}}(n):=\sum_{j}\left(>_{i} N_{k}^{\mathrm{in}}(n)\right.$ are the number of nodes in $G(n)$ with in-degree equal to and strictly greater than , respectively. A similar definition also applies to out-degrees: $N_{j}^{\text {out }}(n):=\sum_{i} N_{i, j}(n)$ and $N_{>j}^{\text {but }}(n):=\sum_{k}\left(>_{j} N_{k}^{\text {out }}(n)\right.$.

It is shown in [3, Theorem 3.2] using concentration inqqualities and martingale methods that for as $n \rightarrow \infty$,

$$
\begin{equation*}
\frac{N_{i, j}(n)}{n} \xrightarrow{P} p_{i j}, \tag{2.13}
\end{equation*}
$$

where $p_{i j}$ is a probability mass function (pmf) and [19, 20, 26] show that $p_{i j}$ is jointly regularly varying and so is the associated joint measure. The analytical form of $p_{i j}$ is given in [3], but later in Section 5.1, we give another proof using Section 4's embedding technique.

From [3, Theorem 3.1], the scaled marginal degree counts $N_{i}^{\text {in }}(n) / n$ and $N_{j}^{\text {out }}(n) / n, i, j \geq$ 0 , also converge:

$$
\begin{equation*}
\frac{N_{0}^{\text {in }}(n)}{n} \xrightarrow{P} p_{0}^{\text {in }}=\frac{\alpha}{1+c_{\mathrm{in}} \delta_{\mathrm{i} n}}, \quad \frac{N_{0}^{\text {out }}(n)}{n} \xrightarrow{P} p_{0}^{\text {out }}=\frac{\gamma}{1+c_{\mathrm{out}} \delta_{\mathrm{out}}}, \tag{2.14}
\end{equation*}
$$

$$
\begin{equation*}
\frac{N_{i}^{\mathrm{in}}(n)}{n} \xrightarrow{P} p_{i}^{\mathrm{in}}=\frac{\Gamma\left(i+\delta_{\mathrm{in}}\right)}{\Gamma\left(i+1+\delta_{\mathrm{in} n}+c_{\mathrm{in}}^{-1}\right)} \frac{\Gamma\left(1+\delta_{\mathrm{in}}+c_{\mathrm{in}}^{-1}\right)}{\Gamma\left(1+\delta_{\mathrm{in} n}\right)}\left(\frac{\alpha \delta_{\mathrm{i} n}}{1+c_{\mathrm{in}} \delta_{\mathrm{i} n}}+\frac{\gamma}{c_{\mathrm{in}}}\right)(i \geq 1 \tag{2.15}
\end{equation*}
$$

$$
\begin{equation*}
\frac{N_{j}^{\text {out }}(n)}{n} \xrightarrow{P} p_{j}^{\text {out }}=\frac{\Gamma\left(j+\delta_{\text {out }}\right)}{\Gamma\left(j+1+\delta_{\text {out }}+c_{\text {out }}^{-1}\right)} \frac{\Gamma\left(1+\delta_{\text {out }}+c_{\text {out }}^{-1}\right)}{\Gamma\left(1+\delta_{\text {out }}\right)}\left(\frac{\gamma \delta_{\text {out }}}{1+c_{\text {out }} \delta_{\text {out }}}+\frac{\alpha}{c_{\text {out }}}\right)(j \geq 1 \tag{2.16}
\end{equation*}
$$

$\operatorname{Both}\left(p_{i}^{\text {in }}\right)_{i}\left\{_{0}\right.$ and $\left(p_{j}^{\text {out }}\right)_{f}\{\geq 0$ are pmf's and the asymptotic form follows from Stirling's formula:

$$
\begin{aligned}
p_{i}^{\text {in }} \sim C_{I N} \cdot i^{-\left(1+c_{\text {in }}^{-1}\right)}, & i \rightarrow \infty \\
p_{j}^{\text {out }} \sim C_{\text {OUT }} \cdot j^{-\left(1+c_{\text {out }}^{-1}\right)}, & j \rightarrow \infty
\end{aligned}
$$

Let $p_{>i}^{\text {in }}=\sum_{k>i} p_{k}^{\text {in }}$ and $p_{>j}^{\text {out }}=\sum_{k>j} p_{k}^{\text {out }}$ be the complementary cdf's and by Scheffé's lemma as well as [22, Equation (8.4.6)], we have

$$
\begin{equation*}
\frac{N_{>j}^{\text {out }}(n)}{n} \xrightarrow{P} p_{>j}^{\text {out }}:=\frac{\Gamma\left(j+1+\delta_{\mathrm{out}}\right)}{\Gamma\left(j+1+\delta_{\mathrm{out}}+c_{\mathrm{out}}^{-1}\right)} c_{\mathrm{out}} \frac{\Gamma\left(1+\delta_{\mathrm{out}}+c_{\mathrm{out}}^{-1}\right)}{\Gamma\left(1+\delta_{\mathrm{out}}\right)}\left(\frac{\gamma \delta_{\mathrm{out}}}{1+c_{\mathrm{out}} \delta_{\mathrm{out}}}+\frac{\alpha}{c_{\mathrm{out}}}\right)( \tag{2.18}
\end{equation*}
$$

so again by Stirling's formula we get from (2.17) and (2.18) that

$$
\begin{aligned}
p_{>i}^{\text {in }} \sim C_{I N}^{\prime} \cdot i^{-c_{\text {in }}^{-1}}=: C_{I N}^{\prime} \cdot i^{-l_{\text {in }}}, & i \rightarrow \infty \\
p_{>j}^{\text {out }} \sim C_{O U T}^{\prime} \cdot j^{-c_{\text {out }}^{-1}}=: C_{O U T}^{\prime} \cdot j^{-\iota_{\text {out }}}, & j \rightarrow \infty .
\end{aligned}
$$

In other words, the marginal tail distributions of the asymptotic in- and out-degree sequences in a directed linear preferential attachment model are asymptotic to power laws with tail indices $\iota_{\text {in }} \equiv c_{\text {in }}^{-1}$ and $\iota_{\text {out }} \equiv c_{\text {out }}^{-1}$, respectively.

## 3. Preliminaries: Switched Birth Immigration Processes.

In this section, we introduce a pair of switched birth immigration processes (SBI processes). This lays the foundation for Section 4, where we embed the in- and out-degree sequences of a fixed network node into a pair of SBI processes and derive the asymptotic limit of the degree growth.
3.1. Birth immigration processes. We start with a brief review of the birth immigration process. A linear birth process with immigration (BI process), $\{Z(t): t \geq 0\}$, having lifetime parameter $\lambda>0$ and immigration parameter $\theta \geq 0$ is a continuous time Markov process with state space $\mathbb{N}=\{0,1,2,3, \ldots\}$ and transition rate

$$
q_{k, k+1}^{Z}=\lambda k+\theta, \quad k \geq 0
$$

When $\theta=0$ there is no immigration and the BI process becomes a pure birth process and in such cases, the process usually starts from 1.

For $\theta>0$, the BI process starting from 0 can be constructed from a Poisson process and an independent family of iid linear birth processes [21]. Suppose that $N_{\theta}(t)$ is the counting function of homogeneous Poisson points $0<\tau_{1}<\tau_{2}<\ldots$ with rate $\theta$ and independent of this Poisson process we have independent copies of a linear birth process $\left\{\zeta_{i}(t): t \geq 0\right\}_{i \geq 1}$ with parameter $\lambda>0$ and $\zeta_{i}(0)=1$ for $i \geq 1$. The BI process $Z(t), t \geq 0$ is a shot noise process with $Z(0)=0$ and for $t \geq 0$,

$$
\begin{equation*}
Z(t):=\sum_{i=1}^{\infty} f_{i}\left(t-\tau_{i}\right) \mathbf{1}_{\left\{t \geq \tau_{i}\right\}}=\sum_{i=1}^{N_{\theta}(t)} \zeta_{i}\left(t-\tau_{i}\right) . \tag{3.1}
\end{equation*}
$$

Theorem 3.1 modifies slightly the statement of [21, Theorem 5] summarizing the asymptotic behavior of the BI process. This is also reviewed in [27].

Theorem 3.1. For $\{Z(t): t \geq 0\}$ as in (3.1), we have as $t \rightarrow \infty$,

$$
\begin{equation*}
e^{-\lambda t} Z(t) \xrightarrow{\text { a.s. }} \sum_{i=1}^{\infty}\left(W_{i} e^{-\lambda \tau_{i}}=: \sigma\right. \tag{3.2}
\end{equation*}
$$

where $\left\{W_{i}: i \geq 1\right\}$ are independent unit exponential random variables satisfying a.s. for each $i \geq 1$,

$$
W_{i}=\lim _{t \rightarrow \infty} e^{-t} \zeta_{i}(t)
$$

The random variable $\sigma$ in (3.2) is a.s. finite and has a Gamma density given by

$$
f(x)=\frac{1}{\Gamma(\theta / \lambda)} x^{\theta / \lambda-1} e^{-x}, \quad x>0
$$

Remark 3.2. The form of $\sigma$ in (3.2) and its Gamma density is justified in [21, 27]. For a BI process $\left\{Z^{\prime}(t)\right\}_{t \geq 0}$ with $Z^{\prime}(0)=j \geq 1$, modifying the representation in (3.1) gives

$$
Z^{\prime}(t)=\sum_{i=1}^{j} \zeta_{i}(t)+\sum_{i=j+1}^{\infty}\left(\zeta_{i}\left(t-\tau_{i}\right) \mathbf{1}_{\left\{t \geq \tau_{i}\right\}}\right.
$$

Therefore, $e^{-\lambda t} Z^{\prime}(t) \xrightarrow{\text { a.s. }} \sigma^{\prime}$ where $\sigma^{\prime}$ has a Gamma density given by $g(x)=x^{j+\theta / \lambda-1} e^{-x} / \Gamma(j+$ $\theta / \lambda), x>0$.

| Process | $I^{(0)}(t)$ | $I^{(1)}(t)$ | $O^{(0)}(t)$ | $O^{(1)}(t)$ |
| :---: | :---: | :---: | :---: | :---: |
| $t=0$ | 0 | 1 | 1 | 0 |
| Rate | $(1-p)\left(k+\delta_{1}\right)$ | $p\left(k+\delta_{2}\right)$ |  |  |

Table 1. Ingredients for a pair of switched BI processes.
3.2. Switched birth immigration processes. A switched birth immigration (SBI) process uses a Bernoulli choice variable to choose randomly from two independent BI processes with the same linear transition rates with one starting from 1 at $t=0$ and the other starting from 0. A pair of SBI processes takes two SBI processes which are linked through the same Bernoulli choice variable.

Suppose that $J$ is a Bernoulli switching random variable with

$$
\mathbf{P}(J=1)=p=1-\mathbf{P}(J=0)
$$

and $\left\{I^{(0)}(t): t \geq 0\right\},\left\{I^{(1)}(t): t \geq 0\right\},\left\{O^{(0)}(t): t \geq 0\right\},\left\{O^{(1)}(t): t \geq 0\right\}$ are four independent BI processes (also independent of $J$ ) with $I^{(0)}(0)=O^{(1)}(0)=0, I^{(1)}(0)=O^{(0)}(0)=1$ and transition rates

$$
\begin{array}{ll}
q_{k, k+1}^{I^{(0)}}=(1-p)\left(k+\delta_{1}\right), & q_{k, k+1}^{O^{(1)}}=p\left(k+\delta_{2}\right), \\
\text { for } k \geq 0 \\
q_{k, k+1}^{I^{(1)}}=(1-p)\left(k+\delta_{1}\right), & q_{k, k+1}^{O(0)}=p\left(k+\delta_{2}\right), \\
\text { for } k \geq 1, \delta_{1}, \delta_{2}>0
\end{array}
$$

See Table 1 for quick reminders. Then we construct a pair of SBI processes $\left\{\left(I^{(J)}(t), O^{(J)}(t)\right) \notin\right.$ $t \geq 0\}$ using five independent ingredients:

$$
\begin{align*}
& \text { We then cdnsider the convergence of the pair of SBI processes, }\left(e^{-(1-p) t} I^{(J)}(t), e^{-p t} O^{(J)}(t)\right), \text {, } \tag{3.3}
\end{align*}
$$ as $t \rightarrow \infty$. Write a Gamma random variable $X$ with density $f_{X}(x)=b^{a} x^{a-1} e^{-b x} / \Gamma(a), x>0$ and $a, b>0$, as $X \sim \Gamma(a, b)$. Then from Theorem 3.1, Remark 3.2 and (3.3), we have with $X^{(0)}, Y^{(0)}, X^{(1)}, Y^{(1)}$ being four independent Gamma random variables and $X^{(0)} \sim \Gamma\left(\delta_{0}, 1\right)$, $Y^{(0)} \sim \Gamma\left(1+\delta_{1}, 1\right), X^{(1)} \sim \Gamma\left(1+\delta_{0}, 1\right), Y^{(1)} \sim \Gamma\left(\delta_{1}, 1\right)$, as $t \rightarrow \infty$,

$$
\begin{align*}
& \left(e^{-(1-p) t} I^{(J)}(t), e^{-p t} O^{(J)}(t)\right)  \tag{3.4}\\
& \left.X^{(J)}, Y^{(J)}\right) \text { has joint density }
\end{align*}{ }^{\text {a.s. }}(1-J)\left(X^{(0)}, Y^{(0)}\right)+J\left(X^{(1)}, Y^{(1)}\right)=:\left(X^{(J)}, Y^{(J)}\right) .
$$

$$
\begin{equation*}
f_{X^{(J)}, Y^{(J)}}(x, y)=(1-p) \frac{x^{\delta_{0}-1} e^{-x}}{\Gamma\left(\delta_{0}\right)} \frac{y^{\delta_{1}} e^{-y}}{\Gamma\left(1+\delta_{1}\right)}+p \frac{x^{\delta_{0}} e^{-x}}{\Gamma\left(1+\delta_{0}\right)} \frac{y^{\delta_{1}-1} e^{-y}}{\Gamma\left(\delta_{1}\right)}, \quad x, y>0 . \tag{3.5}
\end{equation*}
$$

## 4. Embedding Process.

In order to prove the weak convergence of the sequence of empirical measures in (1.6), we need to embed the in- and out-degree sequences $\left\{\left(D_{v}^{\text {in }}(n), D_{v}^{\text {out }}(n)\right),(v \in[n], n \geq 1\}\right.$ into a process constructed from pairs of SBI processes, as specified in Section 3. The embedding idea is proposed in [1] and has been used in [27] to model two different undirected linear preferential attachment models.
4.1. Embedding. Here we discuss how to embed the directed network growth model into a process constructed from an infinite sequence of SBI pairs.
4.1.1. Directed network model and SBI processes. The building blocks of the embedding procedure is an infinite family of independent BI processes

$$
\left\{I I_{1}(t), O_{1}(t), I_{v}^{(0)}(t), I_{v}^{(1)}(t), O_{v}^{(0)}(t), O_{v}^{(1)}(t): v \geq 2, t \geq 0\right.
$$

defined on the same probability space and satisfying:
(i) $\left(I_{1}(0), O_{1}(0)\right)=1,\left(I_{v}^{(0)}(0), O_{v}^{(0)}\right)=(0,1)$ and $\left(I_{v}^{(1)}(0), O_{v}^{(1)}(0)\right)=(1,0)$, for each $v \geq 2$
(ii) Any process labeled with an $I$ is a BI process with transition rates

$$
q_{k, k+1}^{I}=\frac{c_{\mathrm{in}}}{c_{\mathrm{in}}+c_{\mathrm{out}}}\left(k+\delta_{\mathrm{i} n}\right), \quad \delta_{\mathrm{i} n}>0
$$

and any process labeled with an $O$ is a BI process with transition rates

$$
q_{k, k+1}^{O}=\frac{c_{\mathrm{out}}}{c_{\mathrm{in}}+c_{\mathrm{out}}}\left(k+\delta_{\mathrm{out}}\right) \quad \delta_{\mathrm{out}}>0 .
$$

These hold for $k \geq 0$ when $v \geq 2$ and $k \geq 1$ for $I_{1}, O_{1}$.
On $\left(\mathbb{N}^{2}\right)^{\infty}$, define

$$
\boldsymbol{Z}^{(1)}=\left\{\boldsymbol{Z}_{t}^{(1)}: t \geq 0\right\}:=\left\{\left(\left(I_{1}(t), O_{1}(t)\right),(0,0), \ldots\right): t \geq 0\right\}
$$

and the $\sigma$-algebra $\mathcal{F}_{t}^{(1)}:=\sigma\left\{\boldsymbol{Z}_{t}^{(1)}: 0 \leq s \leq t\right\}$ (so that $\boldsymbol{Z}^{(1)}$ is strong Markov with respect to $\left\{\mathcal{F}_{t}^{(1)}\right\}$. Set $T_{1}=0$ and define the stopping the $T_{2}$ with respect to $\left\{\mathcal{F}_{t}^{(1)}, t \geq 0\right\}$ as

$$
\begin{equation*}
T_{2}:=\inf \left\{t \geq 0: \boldsymbol{Z}_{t}^{(1)} \text { jumps }\right\}( \tag{4.1}
\end{equation*}
$$

Then $T_{2}$ is the minimum of two independent exponential r.v. $s$ with means

$$
\left(\frac{c_{\mathrm{in}}}{c_{\mathrm{in}}+c_{\mathrm{out}}}\left(1+\delta_{\mathrm{in} n}\right)\right)^{-1} \quad \text { and } \quad\left(\frac{c_{\mathrm{out}}}{c_{\mathrm{in}}+c_{\mathrm{out}}}\left(1+\delta_{\mathrm{out}}\right)\right)^{-1} .
$$

From (2.3), we have

$$
\mathbf{P}\left[T_{2}>t\right]=e^{-\left(c_{\mathrm{in}}+c_{\mathrm{out}}\right)^{-1} t}, \quad t>0
$$

Let $J_{2}:=\mathbf{1}_{\left\{O_{1} \text { jumps first }\right\}}$ so that $\mathbf{P}\left[J_{2}=1\right]=\gamma$. Also, let $\widetilde{L}_{2}$ be index of the $(I, O)$-pair that jumps first at $T_{2}$ which in this case is 1. However, note that $\left(\widetilde{L}_{2} J_{2}\right)$ determines which one of $I_{1}$ and $O_{1}$ will jump at $T_{2}$, and $T_{2}$ is independent of $\left(\widetilde{L}_{2} J_{2}\right)$ ( $x$ t the property of independent exponential r.v.'s (cf. [17, Exercise 4.45(a)]). In addition, we also have $T_{2}, \widetilde{L}_{2} J_{2} \in \mathcal{F}_{T_{2}}^{(1)}$, that is, measurable with respect to $\mathcal{F}_{T_{2}}^{(1)}$.

Now use the independent quantities $J_{2},\left(I_{2}^{(0)}, O_{2}^{(0)}\right),\left(I_{2}^{(1)}, O_{2}^{(1)}\right)$ to define a pair of SBI processes $\left(I_{2}, O_{2}\right)=\left(\left(I_{2}^{\left(J_{2}\right)}, O_{2}^{\left(J_{2}\right)}\right)\right.$ as in (3.3). Let $\boldsymbol{z}_{2}(t):=\left((0,0),\left(I_{2}^{\left(J_{2}\right)}(t), O_{2}^{\left(J_{2}\right)}(t)\right),(0,0), \ldots\right)$ and

$$
\boldsymbol{Z}^{(2)}=\left\{\boldsymbol{Z}_{t}^{(2)}: t \geq 0\right\}:=\left\{\boldsymbol{Z}_{t+T_{2}}^{(1)}+\boldsymbol{z}_{2}(t): t \geq 0\right\}(
$$

Define the $\sigma$-algebra

$$
\mathcal{F}_{t+T_{2}}^{(2)}:=\sigma\left\{\boldsymbol{母}_{s}^{(2)}: 0 \leq s \leq t\right\} \bigvee \mathcal{F}_{T_{2}}^{(1)}
$$

so that $\boldsymbol{Z}^{(2)}$ is strong Markov with respect to $\left\{\mathcal{F}_{t+T_{2}}^{(2)}, t \geq 0\right\}$. Also, let

$$
\tau_{3}:=\inf \left\{t \geq 0: \boldsymbol{Z}_{t}^{(2)} \text { jumps }\right\}\left(T_{3}:=T_{2}+\tau_{3}\right.
$$

and $J_{3}:=1_{\left\{\text {One of } O_{1}\left(T_{2}+\cdot\right), O_{2}^{\left(J_{2}\right)}(\cdot) \text { jumps first }\right\}}$. Denote the index of the $(I, O)$-pair that jumps at $T_{3}$ by $\widetilde{L}_{3}$ and write $\mathbf{P}^{\mathcal{F}_{T_{2}}^{(1)}}(\cdot):=\mathbf{P}\left(\cdot \mid \mathcal{F}_{T_{2}}^{(1)}\right), \mathbf{P}_{\boldsymbol{z}}\left(\boldsymbol{Z}_{t} \in \cdot\right):=\mathbf{P}\left(\boldsymbol{Z}_{t} \in \cdot \mid \boldsymbol{Z}_{0}=\boldsymbol{z}\right)$. Then by the strong Markov property, we have

$$
\mathbf{P}^{\mathcal{F}_{T_{2}}^{(1)}}\left(\boldsymbol{Z}_{t}^{(2)} \in \cdot\right)=\mathbf{P}_{\boldsymbol{Z}_{T_{2}}^{(1)}+\boldsymbol{z}_{2}(0)}\left(\boldsymbol{\not}_{t}^{(1)}+\boldsymbol{z}_{2}(t) \in \cdot\right)(
$$

Therefore, with respect to $\mathbf{P}^{\mathcal{F}_{T_{2}}^{(1)}}$, $\tau_{3}$ is the minimum of 4 independent exponential r.v.'s with means $\left(\frac{c_{\text {in }}}{c_{\text {in }}+c_{\text {out }}}\left(I_{1}\left(T_{2}\right)+\delta_{\text {in }}\right)\right)^{-1},\left(\frac{c_{\text {out }}}{c_{\text {in }}+c_{\text {out }}}\left(O_{1}\left(T_{2}\right)+\delta_{\text {out }}\right)\right)^{-1},\left(\frac{c_{\text {in }}}{c_{\text {in }}+c_{\text {out }}}\left(J_{2}+\delta_{\text {in }}\right)\right)^{-1}$ and $\left(\frac{c_{\text {out }}}{c_{\text {in }}+c_{\text {out }}}\left(1-J_{2}+\delta_{\text {out }}\right)\right)^{-1}$. Note that $\left(I_{1}\left(T_{2}\right), O_{1}\left(T_{2}\right)\right)=\left(2-J_{2}, 1+J_{2}\right)$. We then have the following:
(1) $\mathbf{P}^{\mathcal{F}_{T_{2}}^{(1)}}\left(\tau_{3}>t\right)=e^{-2\left(c_{\text {in }}+c_{\text {out }}\right)^{-1} t}, t>0$.
(2) $\mathbf{P}^{\mathcal{F}_{T_{2}}^{(1)}}\left(J_{3}=1\right)=\gamma$ and $\tau_{3}$ is independent of $\left(\widetilde{L}_{3} J_{3}\right)$ with respect to $\mathbf{P}^{\mathcal{F}_{T_{2}}^{(1)}}$.
(3) The random variables $T_{3}, \widetilde{L}_{3} J_{3} \in \mathcal{F}_{T_{3}}^{(2)}=\mathcal{F}_{\tau_{3}+\mathcal{T}}^{(2)}$.

Continue in this way to use the fonditionally independent quantities $J_{3},\left(I_{3}^{(0)}, O_{3}^{(0)}\right)$ and $\left(I_{3}^{(1)}, O_{3}^{(1)}\right)$ to define a pair of SBI processes $\left(I_{3}, O_{3}\right)=\left(I_{3}^{\left(J_{3}\right)}, O_{3}^{\left(J_{3}\right)}\right)$ as in (3.3). In general, for $n \geq 3$, set

$$
\boldsymbol{Z}_{t}^{(n)}:=\left(\left(I_{1}\left(T_{n}+t\right), O_{1}\left(T_{n}+t\right)\right),\left(I I_{2}^{\left(J_{2}\right)}\left(T_{n}-T_{2}+t\right), O_{2}^{\left(J_{2}\right)}\left(T_{n}-T_{2}+t\right)\right)\right.
$$

$$
\ldots,\left(I_{n}^{\left(J_{n}\right)}(t), O_{n}^{\left(J_{n}\right)}(t)\right),(0,(0), \ldots)\left(\begin{array}{r}
t \geq 0 \\
n^{(n)}
\end{array}\right.
$$

$\mathcal{F}_{t+T_{n}}^{(n)}:=\sigma\left\{\boldsymbol{\not}_{s}^{(n)}: 0 \leq s \leq t\right\} \bigvee \mathcal{F}_{T_{n}}^{(n-1)}, \tau_{n+1}:=\inf \left\{t \geq 0: \boldsymbol{Z}_{t}^{(n)}\right.$ jumps $\}$ and $T_{n+1}:=T_{n}+$ $\tau_{n+1}$. Also, define

- $J_{n+1}:=\mathbf{1}_{\left\{\text {One of } O_{1}\left(T_{n}+\cdot\right), O_{k}^{\left(J_{k}\right)}\left(T_{n}-T_{k}+\cdot\right), k=2, \ldots, n \text { jumps first }\right\}}$, and
- $\widetilde{L}_{n+1}$ is the index of the $(I, O)$-pair that jumps first among $\left(I_{1}\left(T_{n}+t\right), O_{1}\left(T_{n}+\right.\right.$ t) $\left\{,\left(I_{k}\left(T_{n}-T_{k}+t, O_{k}\left(T_{n}-T_{k}+t\right), k=2, \ldots, n\right.\right.\right.$.

Note that with

$$
\boldsymbol{z}_{n}(t):=((0,0), \ldots, \underbrace{\left(I_{n}^{\left(J_{n}\right)}(t), O_{n}^{\left(J_{n}\right)}(t)\right)}_{{ }^{n \text {-th pfir }}},(0,0), \ldots)(
$$

we have $\boldsymbol{Z}_{t}^{(n)}=\boldsymbol{Z}_{\tau_{n}+t}^{(n-1)}+\boldsymbol{z}_{n}(t)$. Using the strong Markov property gives

$$
\left.\mathbf{P}^{\mathcal{F}_{T_{n}}^{(n-1)}}\left(\boldsymbol{Z}_{t}^{(n)} \in \cdot\right)=\mathbf{P}_{\boldsymbol{Z}_{\tau_{n}}^{(n-1)}+\boldsymbol{z}_{n}(0)} \quad \boldsymbol{Z}_{t}^{(1)}+\sum_{k=2}^{n} \boldsymbol{z}_{k}(t) \in \cdot\right)(
$$

Then with respect to $\mathcal{F}_{T_{n}}^{(n-1)}, \tau_{n+1}$ is the minimum of $2 n$ independent exponential r.v.'s with means

$$
\begin{gathered}
\left(\frac{c_{\mathrm{in}}}{c_{\mathrm{in}}+c_{\mathrm{out}}}\left(I_{1}\left(T_{n}\right)+\delta_{\mathrm{in}}\right)\right)^{-1},\left(\frac{c_{\mathrm{out}}}{c_{\mathrm{in}}+c_{\mathrm{out}}}\left(O_{1}\left(T_{n}\right)+\delta_{\mathrm{out}}\right)\right)^{-1}, \\
\left(\frac{c_{\mathrm{in}}}{c_{\mathrm{in}}+c_{\mathrm{out}}}\left(I_{k}^{\left(J_{k}\right)}\left(T_{n}-T_{k}\right)+\delta_{\mathrm{in}}\right)\right)^{-1},\left(\frac{c_{\mathrm{out}}}{c_{\mathrm{in}}+c_{\mathrm{out}}}\left(O_{k}^{\left(J_{k}\right)}\left(T_{n}-T_{k}\right)+\delta_{\mathrm{out}}\right)\right)^{-1}, k=2, \ldots, n .
\end{gathered}
$$

This implies:
(1) The random variable $\tau_{n+1}$ is independent of $\left(\widetilde{L}_{n+1}, J_{n+1}\right)$ with respect to $\mathbf{P}^{\mathcal{F}_{T_{n}}^{(n-1)}}$.
(2) The random variables $T_{n+1}, \widetilde{L}_{n+1}, J_{n+1} \in \mathcal{F}_{T_{n+1}}^{(n)}$.

Set $\tau_{2}:=T_{2}$. Then from this construction follow properties of the distribution of $\left\{\tau_{n}\right\}_{n \geq 2}$ and $\left\{J_{n}\right\}_{n \geq 2}$.

Lemma 4.1. Suppose $\left\{T_{n}\right\}_{n \geq 1},\left\{\tau_{n}\right\}_{n \geq 2}$ and $\left\{J_{n}\right\}_{n \geq 2}$ are defined as above. Then:
(i) The sequence $\left\{J_{n}\right\}$ is independent of $\left\{\tau_{n}\right\}$.
(ii) The sequence $\left\{J_{n}\right\}$ is a sequence of iid Bernoulli random variables with

$$
\begin{equation*}
\boldsymbol{P}\left(J_{n}=1\right)=\gamma=1-\boldsymbol{P}\left(J_{n}=0\right), \quad n \geq 2 \tag{4.2}
\end{equation*}
$$

(iii) The sequence $\left\{\tau_{n}\right\}_{n \geq 2}$ satisfies

$$
\begin{equation*}
\left\{\tau_{n+1}: n \geq 1\right\} \stackrel{d}{=}\left\{\frac{E_{n}}{\left(c_{i n}+c_{o u t}\right)^{-1} n}, n \geq 1\right\}( \tag{4.3}
\end{equation*}
$$

where $\left\{E_{n}: n \geq 1\right\}$ is a sequence of iid unit exponential random variables. So $\left\{T_{n}\right\}$ are the birth times of a linear birth process with birth rate $\left(c_{i n}+c_{\text {out }}\right)^{-1}$.

Proof. For brevity of notation, write $\lambda_{n}^{I_{1}}=\frac{c_{\text {in }}}{c_{\text {in }}+c_{\text {out }}}\left(I_{1}\left(T_{n}\right)+\delta_{\text {in }}\right), \lambda_{n}^{O_{1}}=\frac{c_{\text {out }}}{c_{\text {in }}+c_{\text {out }}}\left(O_{1}\left(T_{n}\right)+\delta_{\text {out }}\right)$ and for $2 \leq k \leq n, n \geq 2$,

$$
\begin{aligned}
\lambda_{n}^{I_{k}} & =\frac{c_{\mathrm{in}}}{c_{\mathrm{in}}+c_{\mathrm{out}}}\left(I_{k}^{\left(J_{k}\right)}\left(T_{n}-T_{k}\right)+\delta_{\mathrm{i} n}\right), \\
\lambda_{n}^{O_{k}} & =\frac{c_{\mathrm{out}}}{c_{\mathrm{in}}+c_{\mathrm{out}}}\left(O_{k}^{\left(J_{k}\right)}\left(T_{n}-T_{k}\right)+\delta_{\mathrm{out}}\right)
\end{aligned}
$$

At each $T_{n}, n \geq 2$, we start a new pair of SBI processes $\left(I_{n}(\cdot), O_{n}(\cdot)\right)$ with initial value $\left(J_{n}, 1-J_{n}\right)$ and one of $\left(I_{k}(\cdot), O_{k}(\cdot)\right), 1 \leq k \leq n-1$ increases by $\left(1-J_{n}, J_{n}\right)$. This corresponds in the network, for instance if $J_{n}=1$, to one of the existing $n-1$ nodes having an out-degree increase by 1 and a new node $n$ with in-degree 1 and out-degree 0 . Therefore (cf. (2.9)),

$$
\begin{equation*}
I_{1}\left(T_{n}\right)+\sum_{k=2}^{n} I_{k}^{\left(J_{k}\right)}\left(T_{n}-T_{k}\right)=O_{1}\left(T_{n}\right)+\sum_{k=2}^{n} \phi_{k}^{\left(J_{k}\right)}\left(T_{n}-T_{k}\right)=n \tag{4.4}
\end{equation*}
$$

Hence, for $n \geq 2, t_{l}>0$ and $j_{l} \in\{0,1\}$ for $l=2, \ldots, n+1$,

$$
\left.\mathbf{P} \bigcap_{l=2}^{n+1}\left[\tau_{l}>t_{l}, J_{l}=j_{l}\right]\right)=\mathbf{E}\left[\left(\boldsymbol{p}^{\mathcal{F}_{T_{n}}^{(n-1)}} \quad \tau_{n+1}>t_{n+1}, J_{n+1}=j_{n+1}, \bigcap_{l=2}^{n}\left\{\left(\tau_{l}>t_{l}, J_{l}=j_{l}\right)\right](\right.\right.
$$

$$
\begin{equation*}
=\mathbf{E}\left[\mathbf{1}_{\cap_{l=2}^{n}\left\{\tau_{l}>t_{l}, J_{l}=j_{l}\right\}} \mathbf{P}^{\mathcal{F}_{T_{n}}^{(n-1)}}\left(\tau_{n+1}>t_{n+1}, J_{n+1}=j_{n+1}\right)\right]( \tag{4.5}
\end{equation*}
$$

since $\left(\tau_{l}, J_{l}, l=2, \ldots, n\right) \in \mathcal{F}_{T_{n}}^{(n-1)}$. Also, we know that with respect to $\mathbf{P}^{\mathcal{F}_{T_{n}}^{(n-1)}}, \tau_{n+1}$ is the minimum of $2 n$ independent exponential r.v.'s and $J_{n+1}$ is independent of $\tau_{n+1}$. Therefore,

$$
\begin{equation*}
\mathbf{P}^{\mathcal{F}_{T_{n}}^{(n-1)}}\left(\tau_{n+1}>t_{n+1}, J_{n+1}=j_{n+1}\right)=\mathbf{P}^{\mathcal{F}_{T_{n}}^{(n-1)}}\left(\tau_{n+1}>t_{n+1}\right) \mathbf{P}^{\mathcal{F}_{T_{n}}^{(n-1)}}\left(J_{n+1}=j_{n+1}\right) \tag{4.6}
\end{equation*}
$$

Note that
and assuming $j_{n+1}=1$, we have

$$
\begin{align*}
& \mathbf{P}^{\mathcal{F}_{T_{n}}^{(n-1)}\left(\tau_{n+1}>t_{n+1}\right)}=\exp \left\{\left(t_{n+1} \sum_{k=1}^{n}\left(\chi_{n}^{I_{k}}+\lambda_{n}^{O_{k}}\right)\right\}( \right.  \tag{4.7}\\
& =\exp \left\{t_{n+1}\left(c_{\mathrm{in}}+c_{\mathrm{out}}\right)^{-1} n\right.
\end{align*}
$$

$$
\begin{equation*}
\mathbf{P}^{\mathcal{F}_{T_{n}}^{(n-1)}}\left(J_{n+1}=1\right)=\frac{\sum_{n}^{n}(=1}{\lambda_{n}^{O_{k}}} \sum_{k}^{n}=1\left(f_{n}^{I_{k}}+\lambda_{n}^{O_{k}}\right)=\gamma \tag{4.8}
\end{equation*}
$$

So (4.5) becomes (continuing to suppose $j_{n+1}=(1)$,

$$
\left.\mathbf{P} \quad \bigcap_{l=2}^{n+1}\left[\tau_{l}>t_{l}, J_{l}=j_{l}\right]\right)=\gamma \exp \left\{-t_{n+1}\left(c_{\mathrm{in}}+c_{\mathrm{out}}\right)^{-1} n \quad \mathbf{P} \quad \bigcap_{l=2}^{n}\left[\tau_{l}>t_{l}, J_{l}=j_{l}\right]\right)(
$$

If $j_{n+1}=0, \gamma$ is replaced by $\alpha$ on the right side. This is sufficient for the proof of the Lemma.
4.1.2. Embedding. The following embedding theorem is similar to those proved in $[1,27]$ and summarizes how to embed in the paired SBI process constructions.
Theorem 4.2. Suppose that $\left\{T_{n}\right\}_{n \geq 1}$ and $\left\{\boldsymbol{Z}_{t}^{(n)}: t \geq 0\right\}$ are as defined in Section 4.1.1. Then in $\left(\left(\mathbb{N}^{2}\right)^{\infty}\right)^{\infty}$,

$$
\{\mathcal{D}(n), n \geq 1\} \stackrel{d}{=}\left\{\boldsymbol{Z}_{0}^{(n)}, n \geq 1\right\}(
$$

Proof. The proof relies on both $\{\mathcal{D}(n), n \geq 1\}$ and $\left\{\boldsymbol{Z}_{0}^{(n)}, n \geq 1\right\}$ being Markov chains with the same transition probabilities. It is similar to that of [1, Theorem 2.1] and [27, Theorem 2] which we now outline.

Define
Recall that $\tilde{\boldsymbol{d}}_{j}^{\left(J_{n}\right)}:=((0,0), \ldots, \underbrace{\left(1-J_{n}, J_{n}\right)}_{j \text {-th pair }},(0,0), \ldots,(0,0),(\underbrace{\left(J_{n}, 1-J_{n}\right)}_{n \text {-th pair }},(0,0), \ldots)(x$

$$
\begin{equation*}
\boldsymbol{Z}_{0}^{(n+1)}=\boldsymbol{Z}_{0}^{(n)}+\tilde{\boldsymbol{d}}_{\tilde{\mathscr{L}}_{n+1}}^{\left(J_{n+1}\right)} \tag{4.9}
\end{equation*}
$$

This expresses $\boldsymbol{Z}_{0}^{(n+1)}$ as a function of $\mathcal{F}_{T_{n}}^{(n-1)}$-measurable random elements and random elements independent of $\mathcal{F}_{T_{n}}^{(n-1)}$, namely:
(1) $\boldsymbol{Z}_{0}^{(n)} \in \mathcal{F}_{T_{n}}^{(n-1)}$;
(2) $J_{n+1}$ which is independent of $\mathcal{F}_{T_{n}}^{(n-1)}$ (by Lemma 4.1; see (4.8));
(3) $\widetilde{L}_{n \neq 1}$ which is a function of $\left(\lambda_{n}^{I_{k}}+\lambda_{n}^{O_{k}}, k=2, \ldots, n\right) \in \mathcal{F}_{T_{n}}^{(n-1)}$ and conditionally on $\mathcal{F}_{f_{n}^{n-1)}}^{n}, 2 n$ i.i.d exponential r.v.s which are independent of $\mathcal{F}_{T_{n}}^{(n-1)}$.
Hence, both $\{\mathcal{D}(n), n \geq 1\}$ and $\left\{\boldsymbol{Z}_{0}^{(n)}, n \geq 1\right\}$ are Markov on the state space $\left(\mathbb{N}^{2}\right)^{\infty}$.
When $n=1$,

$$
\begin{aligned}
\boldsymbol{Z}_{0}^{(1)} & =\left(\left(I_{1}(0), O_{1}(0)\right),(0,0), \ldots\right) \in((1,1),(0,0), \ldots) \\
& =\left(\left(D_{1}^{\text {in }}(1), D_{1}^{\text {out }}(1)\right),(0,0), \ldots\right)=\boldsymbol{\mathcal { D }}(1)
\end{aligned}
$$

so to prove equality in distribution for any $n$, it suffices to verify that the transition probability from $\boldsymbol{Z}_{0}^{(n)}$ to $\boldsymbol{Z}_{0}^{(n+1)}$ is the same as that from $\boldsymbol{\mathcal { D }}(n)$ to $\boldsymbol{\mathcal { D }}(n+1)$ which is given in (2.10) and (2.11). In the SBI setup, applying Lemma 4.1 gives for any $2 \leq v \leq n$,

$$
\begin{aligned}
\mathbf{P}^{\mathcal{F}_{T_{n}}^{(n-1)}}\left(\boldsymbol{\mathcal { Z }}_{0}^{(n+1)}=\right. & \left.\left.\boldsymbol{Z}_{0}^{(n)}+\boldsymbol{e}_{v}^{\text {in }}+\boldsymbol{e}_{n+1}^{\text {out }}\right)\right)=\mathbf{P}^{\mathcal{F}_{T_{n}}^{(n-1)}}\left(J_{n+1}=0, \widetilde{L}_{n+1}=v\right) \\
= & \frac{\frac{c_{\text {in }}}{c_{\text {in }}+c_{\text {out }}}\left(I_{v}^{\left(J_{v}\right)}\left(T_{n}-T_{v}\right)+\delta_{\text {in }}\right)}{\left(c_{\text {in }}+c_{\text {out }}\right)^{-1} n}=\alpha \frac{I_{v}^{\left(J_{v}\right)}\left(T_{n}-\left(T_{v}\right)+\delta_{\text {in }}\right.}{\left(1+\delta_{\text {in }}\right) n}, \\
\mathbf{P}^{\mathcal{F}_{T_{n}}^{(n-1)}}\left(\boldsymbol{母}_{0}^{(n+1)}=\right. & \left.\left.\boldsymbol{Z}_{0}^{(n)}+\boldsymbol{e}_{n+1}^{\text {in }}+\boldsymbol{e}_{v}^{\text {out }}\right)\right)=\mathbf{P}^{\mathcal{F}_{T_{n}}^{(n-1)}}\left(J_{n+1}=1, \widetilde{L}_{n+1}=v\right)( \\
& \frac{\frac{c_{\text {out }}}{c_{\text {in }}+c_{\text {out }}}\left(O_{v}^{\left(J_{v}\right)}\left(T_{n}-T_{v}\right)+\delta_{\text {out }}\right)}{\left(c_{\text {in }}+c_{\text {out }}\right)^{-1} n}=\gamma \frac{O_{v}^{\left(J_{v}\right)}\left(T_{n}-T_{v}\right)+\delta_{\text {out }}}{\left(1+\delta_{\text {out }}\right) n} .
\end{aligned}
$$

For $2 \leq v \leq n$, this agrees with the transition probabilities in (2.10) and (2.11) respectively; the case for $v=1$ is similar.
4.2. Asymptotic properties. With the embedding technique specified in Section 4.1, the asymptotic behavior of the in- and out-degree growth in a preferential attachment model can be characterized explicitly. These asymptotic properties then help us derive weak convergence of the empirical measure. For brevity of notation, we will write $I_{v}^{\left(J_{v}\right)}, O_{v}^{\left(J_{v}\right)}$ as $I_{v}$, $O_{v}, v \geq 2$, in the rest of this paper.
4.2.1. Convergence of the in- and out-degrees for a fixed node. We first consider the asymptotic behavior of the in- and out-degrees for a fixed node, i.e. $\left(D_{v}^{\text {in }}(n), D_{v}^{\text {out }}(n)\right)$ for a fixed $v$. To do this, we make use of the embedding results in Theorem 4.2, which translates the convergence of the degrees to the setting of $\left\{\left(I v_{v}\left(t-T_{v}\right), O_{v}\left(t-T_{v}\right)\right)\left(t \geq T_{v}{ }_{1 \leq v \leq n}\right.\right.$. Results
are summarized in Theorem 4.3.

Theorem 4.3. Suppose that $\left\{T_{n}: n \geq 1\right\}$ and $\left\{J_{n}: n \geq 2\right\}$ are as defined in Section 4.1.1. Then:
(i) The birth times $\left\{T_{n}\right\}_{n \geq 1}$ satisfy that as $n \rightarrow \infty$,

$$
\begin{equation*}
n \cdot e^{-\left(c_{\text {in }}+c_{o u t}\right)^{-1} T_{n}} \xrightarrow{\text { a.s. }} W \quad \text { and } \quad W \sim \operatorname{Exp}(1) . \tag{4.10}
\end{equation*}
$$

(ii) Let $\left(\sigma_{1}^{\text {in }}, \sigma_{1}^{\text {out }}\right)$ be a pair of independent Gamma random variables with densities

$$
f_{\sigma_{1}^{\text {in }}}(x)=\frac{x^{\delta_{\text {in }}} e^{-x}}{\Gamma\left(1+\delta_{\text {in }}\right)} \quad \text { and } \quad f_{\sigma_{1}^{\text {out }}}(x)=\frac{x^{\delta_{\text {out }}} e^{-x}}{\Gamma\left(1+\delta_{\text {out }}\right)}, x>0, \text { respectively },
$$

and for each $v \geq 2,\left(\sigma_{v}^{i n}, \sigma_{v}^{\text {out }}\right)$ have joint density

$$
\begin{equation*}
f_{\left(\sigma_{v}^{\text {in } \left., \sigma_{v}^{\text {out }}\right)}\right.}(x, y)=\alpha \frac{x^{\delta_{\text {in }}-1} e^{-}}{\Gamma\left(\delta_{\text {in }}\right)} \frac{y^{\delta_{\text {out }}} e^{-y}}{\Gamma\left(1+\delta_{\text {out }}\right)}+\gamma \frac{x^{\delta_{\text {in }}} e^{-x}}{\Gamma\left(1+\delta_{\text {in }}\right)} \frac{y^{\delta_{\text {out }}-1} e^{-y}}{\Gamma\left(\delta_{\text {out }}\right)}, \quad x, y>0 . \tag{4.11}
\end{equation*}
$$

Then for a fixed $v \geq 1$, we have, with $W$ defined as in (4.10),

$$
\begin{equation*}
\left(\frac{D_{v}^{\text {in }}(n)}{n^{c_{i n}}}, \frac{D_{v}^{\text {out }}(n)}{n^{c_{\text {out }}}}\right)\left(\Rightarrow \quad \frac{\sigma_{v}^{\text {in }} e^{-\frac{c_{i n}}{c_{i n}+c_{\text {out }}} T_{v}}}{W^{c_{\text {in }}}}, \frac{\sigma_{v}^{\text {out }} e^{-\frac{c_{\text {out }}}{c_{i n}+c_{\text {out }}} T_{v}}}{W^{c_{o u t}}}\right) \quad n \rightarrow \infty . \tag{4.12}
\end{equation*}
$$

Also, setting $D_{v}^{\text {in }}(n)=0=D_{v}^{\text {out }}(n)$ for all $v \geq n+1$, we get as $n \rightarrow \infty$,

$$
\begin{equation*}
\left(\max _{v \geq 1} \frac{D_{v}^{\text {in }}(n)}{n^{c_{\text {in }}}}, \max _{v \geq 1} \frac{D_{v}^{\text {out }}(n)}{n^{c_{o u t}}}\right)\left(\Rightarrow \max _{v \geq 1} \frac{\sigma_{v}^{\text {in }} e^{-\frac{c_{\text {in }}}{c_{\text {in }}+c_{\text {out }}} T_{v}}}{W^{c_{\text {in }}}}, \max _{v \geq 1} \frac{\sigma_{v}^{\text {out }} e^{-\frac{c_{\text {out }}}{c_{\text {in }}+c_{\text {out }}} T_{v}}}{W^{c_{\text {out }}}}\right) \tag{4.13}
\end{equation*}
$$

Here $T_{v},\left(\sigma_{v}^{\text {in }}, \sigma_{v}^{\text {out }}\right)$ and $W$ are independent for all $v \geq 2$.
Remark 4.4. According to the embedding results in Theorem 4.2, (4.12) also implies that there exists random variables $D_{v}^{(1)}, D_{v}^{(2)}, v \geq 1$, on the space of $\left(D_{v}^{\text {in }}(n), D_{v}^{\text {out }}(n)\right)_{v \geq 1}$ satisfying $D_{v}^{(1)} \stackrel{d}{=} W^{-c_{\text {in }}} \sigma_{v}^{\text {in }} e^{-\frac{c_{\text {in }}}{c_{\text {in }}+c_{\text {out }}} T_{v}}$ and $D_{v}^{(2)} \stackrel{d}{=} W^{-c_{\text {out }}} \sigma_{v}^{\text {out }} e^{-\frac{c_{\text {out }}}{c_{\text {in }}+c_{\text {out }}} T_{v}}, v \geq 1$, such that as $n \rightarrow \infty$,

$$
\left(\frac{D_{v}^{\text {in }}(n)}{n^{c_{\text {in }}}}, \frac{D_{v}^{\text {out }}(n)}{n^{c_{\text {out }}}}\right)\left(\xrightarrow{\text { a.s. }}\left(\not \varnothing_{v}^{(1)}, D_{v}^{(2)}\right)\right.
$$

Proof. (i) From Lemma 4.1(i), $\left\{T_{n}: n \geq 1\right\}$ are jump times of a pure birth process starting from 1 and transition rate

$$
q_{j, j+1}=\left(c_{\mathrm{in}}+c_{\text {out }}\right)^{-1} j, \quad j \geq 1 .
$$

Therefore, (4.10) follows from applying the known convergence results of linear birth processes; see [17, Theorem 5.11.4] and [11, 28], among other sources.
(ii) By Theorem 4.2, to show (4.12), it suffices to show that as $n \rightarrow \infty$,

$$
\begin{equation*}
\left(\frac{I_{v}\left(T_{n}-T_{v}\right)}{n^{c_{\mathrm{in}}}}, \frac{O_{v}\left(T_{n}-T_{v}\right)}{n^{c_{\text {out }}}}\right)\left(\xrightarrow{\text { a.s. }} \quad \frac{\sigma_{v}^{\text {in }} e^{-\frac{c_{\mathrm{in}}}{c_{\text {in }}+c_{\text {out }}} T_{v}}}{W^{c_{\text {in }}}}, \frac{\sigma_{v}^{\text {out }} e^{-\frac{c_{\text {out }}}{c_{\mathrm{in}}+c_{\text {out }}} T_{v}}}{W^{c_{\text {out }}}}\right), . \tag{4.14}
\end{equation*}
$$

With (4.10) available, we prove (4.14) by showing the convergence of

$$
\left(e^{-\frac{c_{\text {in }}}{c_{\text {in }}+c_{\text {out }}}\left(t-T_{v}\right)} I_{v}\left(t-T_{v}\right), e^{-\frac{c_{\text {out }}}{c_{\text {in }}+c_{\text {out }}}\left(t-T_{v}\right)} O_{v}\left(t-T_{v}\right)\right)(
$$

as $t \rightarrow \infty$. According to the construction of the processes $\left\{\left(I I_{v}\left(t-T_{v}\right), O_{v}\left(t-T_{v}\right): t \geq\right.\right.$ $\left.\left.T_{v}\right)\right\}_{(v \geq 1}$, we know that $\left(I_{1}(0), O_{1}(0)\right)=(1,1)$. Then applying the convergence result of a BI prodess in Remark 3.2, we have for independent $\left(\sigma_{1}^{\text {in }}, \sigma_{1}^{\text {out }}\right) \sim\left(\Gamma\left(1+\delta_{\text {in }}, 1\right), \Gamma\left(1+\delta_{\text {out }}, 1\right)\right)$,

$$
\left(e^{-\frac{c_{\text {in }}}{c_{\text {in }}+c_{\text {out }}} t} I_{1}(t), e^{-\frac{c_{\text {out }}}{c_{\text {in }}+c_{\text {out }}} t} O_{v}(t)\right)\left({ }^{\text {a.s. }}\left(\sigma_{1}^{\text {in }}, \sigma_{1}^{\text {out }}\right), \quad t \rightarrow \infty .\right.
$$

Moreover, it follows from (3.4) and (3.5) that

$$
\begin{equation*}
\left(e^{-\frac{c_{\text {in }}}{c_{\text {in }}+c_{\text {out }}}}\left(t-T_{v}\right) I_{v}\left(t-T_{v}\right), e^{-\frac{c_{\text {out }}}{c_{\text {in }}+c_{\text {out }}}\left(t-T_{v}\right)} O_{v}\left(t-T_{v}\right)\right) \xrightarrow{\text { a.s. }}\left(\sigma_{v}^{\text {in }}, \sigma_{v}^{\text {out }}\right), \quad t \rightarrow \infty, \tag{4.15}
\end{equation*}
$$

with $\sigma_{v}^{\text {in }}$ and $\sigma_{v}^{\text {out }}$ having the joint density as in (4.11).
Replacing $t$ with $T_{n}$ in (4.15) gives

$$
\begin{equation*}
\left(\frac{I_{v}\left(T_{n}-T_{v}\right)}{e^{\frac{c_{\text {in }}}{c_{\text {in }}+c_{\text {out }}} T_{n}}}, \frac{O_{v}\left(T_{n}-T_{v}\right)}{e^{\frac{c_{\text {out }}}{c_{\text {in }}+c_{\text {out }}} T_{n}}}\right)\left(\xrightarrow{\text { a.s. }}\left(\oint_{v}^{\text {in }} e^{-\frac{c_{\text {in }}}{c_{\text {in }}+c_{\text {out }}} T_{v}}, \sigma_{v}^{\text {out }} e^{-\frac{c_{\text {out }}}{c_{\text {in }}+c_{\text {out }}} T_{v}}\right)(\text { as } n \rightarrow \infty .\right. \tag{4.16}
\end{equation*}
$$

Therefore, combining (4.10) and (4.16) gives (4.12). For $v \geq 2$, the independence of ( $\left.\sigma_{v}^{\text {in }}, \sigma_{v}^{\text {out }}\right)$ and $T_{v}$ follows from the construction and the independence from $W$ follows from [17, p. 443]; this completes the proof of (4.14).
(iii) We verify (4.13) by showing that as $n \rightarrow \infty$,

$$
\begin{equation*}
\left(\max _{v \geq 1} \frac{I_{v}\left(T_{n}-T_{v}\right)}{e^{\frac{c_{\text {in }}}{c_{\text {in }}+c_{\text {out }}} T_{n}}}, \max _{v \geq 1} \frac{O_{v}\left(T_{n}-T_{v}\right)}{e^{\frac{c_{\text {out }}}{c_{\text {in }}+c_{\text {out }}} T_{n}}}\right) \xrightarrow{\text { a.s. }}\left(\max _{v \geq 1} \sigma_{v}^{\text {in }} e^{-\frac{c_{\text {in }}}{c_{\text {in }}+c_{\text {out }}} T_{v}}, \max _{v \geq 1} \sigma_{v}^{\text {out }} e^{-\frac{c_{\text {out }}}{c_{\text {in }}+c_{\text {out }}} T_{v}}\right)( \tag{4.17}
\end{equation*}
$$

Then combining (4.17) with (4.10) gives the result. We use the proof machinery in 1 , Proposition 3.1] to show (4.17), which is summarized in the following lemma.

Lemma 4.5. Let $a_{n, i}: 1 \leq i \leq n_{n \geq 1}$ be a double array of non-negative numbers such that
(1) For all $i \geq 1, \lim _{n \rightarrow \infty} a_{n, i}=a_{i}<\infty$,
(2) $\sup _{n \geq 1} a_{n, i} \leq b_{i}<\infty$ and
(3) $\lim _{i \rightarrow \infty} b_{i}=0$.

Then $\max _{1 \leq i \leq n} a_{n, i} \rightarrow \max _{i \geq 1} a_{i}$, as $n \rightarrow \infty$.
First note that for each $v \geq 1$,

$$
\begin{aligned}
& \qquad I_{v}\left(T_{n}-T_{v}\right) e^{-\frac{c_{\text {in }}}{c_{\text {in }}+c_{\text {out }}}\left(T_{n}-T_{v}\right)} \leq \sup _{t \geq 0} I_{v}(t) e^{-\frac{c_{\text {in }}}{c_{\text {in }}+c_{\text {out }}} t}=: \widetilde{I}_{v} \\
& \qquad O_{v}\left(T_{n}-T_{v}\right) e^{-\frac{c_{\text {out }}}{c_{\text {in }}+c_{\text {out }}}\left(T_{n}-T_{v}\right)} \leq \sup _{t \geq 0} O_{v}(t) e^{-\frac{c_{\text {out }}}{c_{\text {in }}+c_{\text {out }}} t}=: d_{v} \\
& \text { Let } a_{n, v}^{I}:=I_{v}\left(T_{n}-T_{v}\right) e^{-\frac{c_{\text {in }}}{c_{\text {in }}+c_{\text {out }}} T_{n}}, a_{n, v}^{O}:=O_{v}\left(T_{n}-T_{v}\right) e^{-\frac{c_{\text {out }}}{c_{\text {in }}+c_{\text {out }}} T_{n}} \text { for }(1 \leq v \leq n \text {, and } \\
& b_{v}^{I}:=\widetilde{I}_{v} q^{-\frac{c_{\text {in }}}{c_{\text {in }}+c_{\text {out }}} T_{v}}, b_{v}^{O}:=\widetilde{O}_{v} q^{-\frac{c_{\text {out }}}{c_{\text {in }}+c_{\text {out }}} T_{v}} \text { for } v \geq 1 \text {. Then Lemma 4.5(1) is satisfied by (4.16). } \\
& \text { Also, for each } v \geq 1, \sup _{n \geq} a_{n, v}^{I} \leq b_{v}^{I} \text { and } \sup _{n \geq 1}^{O} a_{n, v}^{O} \leq b_{v}^{O} \text {, which satisfies the criterion in }
\end{aligned}
$$

Following the proof of [1, Theorem 1.1], we check the condition in Lemma 4.5(3) by proving the claim that almost surely, for all $\epsilon>0$,

$$
\begin{equation*}
\widetilde{I}_{v} \leq \epsilon v^{c_{\mathrm{in}}}, \quad \text { and } \quad \widetilde{O}_{v} \leq \epsilon v^{c_{\mathrm{out}}}, \quad \text { for all large } v \tag{4.18}
\end{equation*}
$$

Then as $\epsilon$ is arbitrary, it follows from (4.10) that $b_{v}^{I} \rightarrow 0$ and $b_{v}^{O} \rightarrow 0$ a.s. as $v \rightarrow \infty$. This completes checking the three criteria in Lemma 4.5 and therefore leads to (4.13).

To show (4.18), we use Markov's inequality: for any $r, r^{\prime}>0$ and $v \geq 2$,

$$
\begin{aligned}
\mathbf{P}\left(\widetilde{I}_{v} \beta \epsilon v^{c_{\mathrm{in}}}\right) & \leq \mathbf{E}\left(\widetilde{I}_{2}^{r}\right) /\left(\epsilon^{r} v^{r c_{\mathrm{in}}}\right), \\
\mathbf{P}\left(\widetilde{O}_{v}\left(\notin \epsilon v^{c_{\mathrm{out}}}\right)\right. & \leq \mathbf{E}\left(\widetilde{O}^{\prime}\right) /\left(\epsilon^{r^{\prime}} v^{r^{\prime} c_{\mathrm{out}}}\right),
\end{aligned}
$$

since $I_{v}, O_{v}, v \geq 2$ are iid SBI processes. Hence, if we have

$$
\begin{equation*}
\mathbf{E}\left(\widetilde{I}_{2}^{r}\right)<\infty \text { and } \mathbf{E}\left(\widetilde{O}_{2}^{r}\right)<\infty, \quad \text { for } r>c_{\mathrm{in}}^{-1}, r^{\prime}>c_{\mathrm{out}}^{-1}, \text { respectively, } \tag{4.19}
\end{equation*}
$$

then by Borel-Cantelli, the claim in (4.18) is justified. To prove (4.19), let
then by the construction of $\left(I_{2}(\cdot), O_{2}(\cdot)\right)$, we have

$$
\begin{aligned}
& \mathbf{E}\left(\widetilde{I_{2}^{r}}\right)=\alpha \mathbf{E}\left(T\left(\bar{\varphi}_{2}^{r}\right)+\gamma \mathbf{E}\left(\widetilde{I^{(1)}}{ }_{2}^{r}\right)<\infty,\right. \\
& \mathbf{E}\left(\widetilde{I_{2}^{\prime}}\right)=\alpha \mathbf{E}\left(\widetilde{O^{(0)}}{ }_{2}^{r^{\prime}}\right)+\gamma \mathbf{E}\left(\widetilde{O^{(1)}}{ }_{2}^{r^{\prime}}\right)<\infty,
\end{aligned}
$$

using the assumption that $I_{2}^{(0)},\left\{_{2}^{(1)}, O_{2}^{(0)}\right.$ and $O_{2}^{(1)}$ are independent BI processes so that results in [1, Proposition 2.6] are still applicable here. This completes the proof of (4.17).

## 5. Convergence Results on Joint Degree Distributions.

5.1. Convergence of the joint degree counts. Now we analyze the convergence of the joint empirical distribution of the in- and out-degrees $\left\{\left(D_{v}^{\text {in }}(n), D_{v}^{\text {out }}(n)\right) \div v \in[n]\right\}$, using the SBI embedding technique. Let $B(a, p)$ be a negative binomial integer valued random variable with parameters $a>0$ and $p \in(0,1)$ (abbreviated as $N B(a, p)$ ), and the generating function of $B(a, p)$ is

$$
\mathbf{E}\left(s^{B(a, p)}\right) \hat{=} p^{a}(1-(1-p) s)^{-a}, \quad 0 \leq s \leq 1
$$

We also use the notation $B(a, Z)$ to represent a r.v. having a mixture distribution such that the second parameter of the negative binomial r.v. is randomized by an independent r.v. $Z$.

Theorem 5.1. Let $N_{i, j}(n)$ be the number of nodes with in-degree $i$ and out-degree $j$ in graph $G(n)$, then we have

$$
\begin{equation*}
\frac{N_{i, j}(n)}{n} \xrightarrow{P} \boldsymbol{P}((\mathcal{I}, \mathcal{O})=(i, j)),(\quad \text { as } n \rightarrow \infty \tag{5.1}
\end{equation*}
$$

The limit pair $(\mathcal{I}, \mathcal{O})$ can be represented in distributidn as:

$$
\begin{equation*}
(\mathcal{I}, \mathcal{O}) \stackrel{d}{=}(1-J)\left(X_{1}, 1+Y_{1}\right)+J\left(1+X_{2}, Y_{2}\right) \tag{5.2}
\end{equation*}
$$

where
(i) $J$ is a Bernoulli switching variable with $\boldsymbol{P}(J=1)=1-\boldsymbol{P}(J=0)=\gamma$.
(ii) Suppose $\left\{B^{(1)}\left(\delta_{1}, p\right): p \in(0,1)\right\}$, $\left\{B^{(2)}\left(\delta_{1}^{\prime}, p\right): p \in(0,1)\right\}$, $\left\{\widetilde{B}^{(1)}\left(\delta_{2}, p\right): p \in(0,1)\right\}$ and $\left\{\widetilde{B}^{(2)}\left(\delta_{2}^{\prime}, p\right): p \in(0,1)\right\}, \delta_{1}, \delta_{1}^{\prime}, \delta_{2}, \delta_{2}^{\prime}>0$, are four independent families of negative binomial variables, then

$$
\left.\begin{array}{l}
\left(X_{1}, Y_{1}\right)=\left(B^{(1)}\left(\oint_{\text {in }}, e^{-c_{i n} T}\right)\left(\widetilde{B}^{(1)}\left(1+\delta_{\text {out }}, e^{-c_{\text {out }} T}\right)\right)\right.  \tag{5.3a}\\
\left(X_{2}, Y_{2}\right)=\left(\not \beta^{(2)}\left(1 \gamma^{\left(\delta_{\text {in }}, e^{-c_{\text {in }} T}\right)}\right)_{\left.\widetilde{B}^{(2)}\left(\delta_{\text {out }}, e^{-c_{\text {out }} T}\right)\right)}\right.
\end{array}\right\}
$$

with $T$ being an exponential ra风dom variable with unit mean, independent of $J, B^{(1)}$, $B^{(2)}, \widetilde{B}^{(1)}$ and $\widetilde{B}^{(2)}$.

Remark 5.2. Theorem 5.1 coincides with the known results proven in [19, 20], since $e^{c_{\text {in }} T}$ is a Pareto random variable on $[1, \infty)$ with index $c_{\mathrm{in}}^{-1}$, denoted by $Z$, and $e^{c_{\text {out }} T}=Z^{a}$, with $a:=c_{\mathrm{out}} / c_{\mathrm{in}}$.

Proof. The proof of [25, Lemma 3.1] verifies that

$$
\frac{N_{i, j}(n)}{n}-\frac{\mathbf{E}\left(N_{i, j}(n)\right)}{n} \xrightarrow{P} 0, \quad \text { as } n \rightarrow \infty .
$$

Hence, we are left to examine the difference $\left|\mathbf{E}\left(N_{i, j}(n)\right) / n-\mathbf{P}((\mathcal{I}, \mathcal{O})=(i, j))\right|$. By the embedding results in Theorem 4.2, we have

Suppose that $\left\{B_{v}^{(1)}\left(\delta_{\text {in }}, p\right): v \geq 1\right\},\left\{B_{v}^{(2)}\left(1+\delta_{\text {in }}, p\right): v \geq 1\right\},\left\{\widetilde{B}_{v}^{(1)}\left(1+\delta_{\text {out }}, p\right): v \geq 1\right\}$ and $\left\{\widetilde{B}_{v}^{(2)}\left(\delta_{\text {out }}, p\right): v \geq 1\right\}$ are four independent sequences of negatife binomial r.v.'s with given perameters. Then by the distribution of a BI process (cf. [21, Equation (2.2)] and [8, Theoren( 3.11]), we have for any $v \geq 2, t \geq 0$ and $k \geq 0$,

$$
\begin{align*}
& \mathbf{P}\left(I_{v}^{(0)}(t)=k\right)=\mathbf{P}\left[B_{v}^{(1)}\left(\not_{\text {in }}, e^{-\frac{c_{\text {in }}}{c_{\text {in }}+c_{\text {out }}} t}\right)=k\right],  \tag{5.5a}\\
& \mathbf{P}\left(I_{v}^{(1)}(t)=k\right)=\mathbf{P}\left[1+B_{v}^{(2)}\left(f^{2}+\delta_{\text {in }}, e^{-\frac{c_{\text {in }}}{c_{\text {in }}+c_{\text {out }}} t}\right)=k\right], \tag{5.5b}
\end{align*}
$$

$$
\begin{align*}
& \mathbf{P}\left(O_{v}^{(1)}(t)=k\right)=\mathbf{P}\left[\widetilde{B}_{v}^{(x)}\left(\oint_{\text {out }}, e^{-\frac{c_{\text {out }}}{c_{\text {in }}+c_{\text {out }}} t}\right)=k\right]( \tag{5.5c}
\end{align*}
$$

and note the quantities on the right do not qepend on $v$. Also, recall that $\left(I_{v}(t), O_{v}(t)\right)^{2}(\geq 2$,
$t \geq 0$, are identically distributed such that,

$$
I_{v}(t)=\left(1-J_{v}\right) I_{v}^{(0)}(t)+J_{v} I_{v}^{(1)}(t), \quad O_{v}(t)=\left(1-J_{v}\right) O_{v}^{(0)}(t)+J_{v} O_{v}^{(1)}(t)
$$

Since for $v \geq 2$, the processes $I_{v}^{(0)}, I_{v}^{(1)}, O_{v}^{(0)}$ and $O_{v}^{(1)}$ are independent from each other, we then define for any $v \geq 2$,
and (5.4) becomes,

$$
\left.\begin{array}{l}
\mathcal{B}_{v}^{(n)}:=\left(\left(\left\{1-J_{v}\right) B_{v}^{(1)}\left(\delta_{\mathrm{i} n}, e^{-\left(T_{n}-T_{v}\right)}\right)+J_{v}\left(1+B_{v}^{(2)}\left(1+\delta_{\mathrm{i} n}, e^{-\left(T_{n}-T_{v}\right)}\right),\right.\right.\right. \\
\left(1-J_{v}\right)\left(1+\widetilde{B}_{v}^{(1)}\left(1+\delta_{\mathrm{out}}, e^{-\left(T_{n}-T_{v}\right)}\right)+J_{v}\left(\widetilde{B}_{v}^{(2)}\left(\delta_{\mathrm{out}}, e^{-\left(T_{n}-T_{v}\right)}\right),( \right.\right.
\end{array}\right)
$$

$$
\begin{align*}
& \frac{1}{n} \mathbf{E}\left(N_{i, j}(n)\right)=\frac{1}{n} \sum_{v=1}^{n} \mathbf{P}\left[\left(I / v\left(T_{n}-T_{v}\right), O_{v}\left(T_{n}-T_{v}\right)\right)=(i, j)\right] \\
= & \frac{1}{n} \sum_{v=1}^{n} \mathbf{P}\left[\not \boldsymbol{\phi}_{v}^{(n)}=(i, j)\right] \mathbb{+} \frac{1}{n}\left(\mathbf{P}\left[\left(I /\left(T_{n}\right), O_{1}\left(T_{n}\right)\right)=(i, j)\right]-\mathbf{P}\left[\dot{\&}_{1}^{(n)}=(i, j)\right]\right)( \tag{5.6}
\end{align*}
$$

The last step is necessitated by the construction since $\left(I_{1}(t), O_{1}(t)\right)$ is a pair of independent BI processes, which is different from the rest of the $\left(I_{v}(\cdot), O_{v}(\cdot)\right)_{v \geq 2}$ pairs. Here this difference is inconsequential because as $n \rightarrow \infty$,

$$
\frac{1}{n} \mathbf{P}\left[\left(I I_{1}\left(T_{n}\right), O_{1}\left(T_{n}\right)\right)=(i, j)\right]\left(\mathbf{P}\left[\mathcal{B}_{1}^{(n)}=(i, j)\right] \leq \frac{2}{n} \rightarrow 0\right.
$$

So we only need to consider the first term in (5.6). Let $U_{n}$ be a random variable uniformly distributed on $[n-1]$ and independent of the rest. Then

$$
\begin{aligned}
& \frac{1}{n} \sum_{v=1}^{n} \mathbf{P}\left[\boldsymbol{q}_{v}^{(n)}=(i, j)\right](
\end{aligned}
$$

since the distributions of $B_{v}^{(1)}, \widetilde{B}_{v}^{(1)}, B_{v}^{(2)}, \widetilde{B}_{v}^{(1)}$ do not depend on $v$. Let $T$ be a unit exponential random variable that is independent of $I_{v}, O_{v}, v \geq 1$. A variant of the Renyi representation for exponential order statistics (see [8, Theorem 3.14] for details) gives

$$
\begin{equation*}
T_{n}-T_{U_{n}} \stackrel{d}{=} \frac{T}{\left(c_{\mathrm{in}}+c_{\mathrm{out}}\right)^{-1}} \tag{5.7}
\end{equation*}
$$

Define a Bernoulli random variable $J$ that is independent from $T, B_{1}^{(1)}, B_{1}^{(2)}, \widetilde{B}_{1}^{(1)}$ and $\widetilde{B}_{1}^{(2)}$ with $\mathbf{P}(J=1)=\gamma=1-\mathbf{P}(J=0)$. Then applying (5.7) therefore gives

$$
\begin{aligned}
& \frac{1}{n} \sum_{v=1}^{n} \mathbf{P}\left[\not \boldsymbol{\phi}_{v}^{(n)}=(i, j)\right] \\
& =\alpha\left(1-\frac{1}{n}\right) \boldsymbol{P}\left[\left(\not \phi_{1}^{(1)}\left(\delta_{\mathrm{i} n}, e^{-c_{\mathrm{in}} T}\right), 1+\widetilde{B}_{1}^{(1)}\left(1+\delta_{\mathrm{out}}, e^{-c_{\mathrm{out}} T}\right)\right)=(i, j)\right] \\
& +\gamma\left(1-\frac{1}{n}\right) \boldsymbol{p}\left[\left(1 \left(+B_{1}^{(2)}\left(1+\delta_{\text {in }}, e^{-c_{\mathrm{i}},}(T), \widetilde{B}_{1}^{(2)}\left(\delta_{\text {out }}, e^{-c_{\text {out }} T}\right)=(i, j)\right]\left(+\frac{1}{n}\left[\mathcal{B}_{n}^{(n)}=(i, j)\right]\right.\right.\right.\right. \\
& =\left(1-\frac{1}{n}\right) \boldsymbol{p}\left[( 1 - J ) \left(B_{1}^{(1)}\left(\delta_{\text {in }}, e^{-c_{\text {in }} T}\right), 1+\widetilde{B}_{1}^{(1)}\left(\left(1+\delta_{\text {out }}, e^{-c_{\text {out }} T}\right)\right)\right.\right. \\
& +J\left(1+B_{1}^{(2)}\left(1+\delta_{\text {in }}, e^{-c_{\text {in }} T}\right), \widetilde{B}_{1}^{(2)}\left(\delta_{\text {out }}, e^{\left(c_{\text {out }} T\right.}\right) \neq(i, j)\right]+\frac{1}{n}\left[\mathcal{B}_{n}^{(n)}=(i, j)\right] \\
& =\left(1-\frac{1}{n}\right) \boldsymbol{P}[(\mathcal{I}, \mathcal{O})=(i, j)]+\frac{1}{n}\left[\boldsymbol{申}_{n}^{(n)}=(i, j)\right]( \\
& \text { Therefore, }
\end{aligned}
$$

$$
\begin{aligned}
& \frac{1}{n} \mathbf{E}\left[N_{i j}(n)\right]-\mathbf{P}[(\mathcal{I}, \mathcal{O}) \neq(i, j)]\left(\leq \frac{4}{n},\right. \\
& 5.3) \text { as } n \rightarrow \infty .
\end{aligned}
$$

Remark 5.3. This argument also shows that for $x>0, y>0$,
where

$$
\begin{equation*}
\frac{1}{n} \mathbf{E} N_{>x,>y}(n)=\mathbf{P}((\mathcal{I}, \mathcal{O}) \in(x, \infty] \times(y, \infty])\left(\epsilon_{n}(x, y)\right. \tag{5.8}
\end{equation*}
$$

$$
\sup _{x>0, y>0}\left|\epsilon_{n}(x, y)\right| \leq \frac{4}{n}
$$

5.2. Convergence of the joint empirical measure. In this section, we investigate the convergence of the joint empirical measure:

$$
\frac{1}{k_{n}} \sum_{k=1}^{n} \epsilon_{\left(D_{i}^{\text {in }}(n) / b_{1}\left(n / k_{n}\right), D_{i}^{\text {out }}(n) / b_{2}\left(n / k_{n}\right)\right)}(\cdot),
$$

with scaling functions $b_{i}(\cdot), i=1,2$, and some intermediate sequence $k_{n}$ such that $k_{n} / n \rightarrow 0$ and $k_{n} \rightarrow \infty$ as $n \rightarrow \infty$. From (5.1), we have

$$
\begin{equation*}
\left.\frac{1}{n} \sum_{v \in[n]} \xi_{\left(\phi_{v}^{\text {in }}(n), D_{v}^{\text {out }}(n)\right)}(\{(i, j)\})\right)^{P} \mathbf{P}((\mathcal{I}, \mathcal{O})=(i, j)),(\quad n \rightarrow \infty . \tag{5.9}
\end{equation*}
$$

Moreover, [20, Theorem 2] shows that the limit pair $(\mathcal{I}, \mathcal{O})$ is non-standard regularly varying, i.e.

$$
\begin{equation*}
n \mathbf{P}\left[\left(\frac{\mathcal{I}}{k^{c_{\text {in }}}}, \frac{\mathcal{O}}{n^{c_{\text {out }}}}\right) \in \cdot\right]\left(\stackrel{v}{\longrightarrow} \gamma V_{1}(\cdot)+\alpha V_{2}(\cdot), \quad n \rightarrow \infty\right. \tag{5.10}
\end{equation*}
$$

in $M_{+}\left([0, \infty]^{2} \backslash\{\mathbf{0}\}\right)$ and $V_{i}(\cdot), i=1,2$, concentrate on $(0, \infty)^{2}$ with Lebesgue densities given below in (5.14) and (5.15). It is also shown in [26] that the density of the limit measure is jointly regularly varying, and the relationship between the regular variation of the limit measure and that of the limit density has been explored.

Let $b_{1}(t)=t^{c_{\text {in }}}$ and $b_{2}(t)=t^{c_{\text {out }}}$, then heuristically, combining (5.9) and (5.10) gives

$$
\begin{align*}
\frac{1}{k_{n}} \sum_{v \in[n]} \epsilon_{\left(D_{v}^{\text {in }}(n) /\left(n / k_{n}\right)_{\text {in }}^{c^{\text {in }}} D_{v}^{\text {out }}(n) /\left(n / k_{n}\right)^{c_{\text {out }}}\right)}(\cdot) & \approx \frac{n}{k_{n}} \mathbf{P}\left[\left(\frac{\mathcal{I}}{\left(n / k_{n}\right)^{c_{\text {in }}}}, \frac{\mathcal{O}}{\left(n / k_{n}\right)^{c_{\text {out }}}}\right) \in \cdot\right](  \tag{5.11}\\
& \Rightarrow \gamma V_{1}(\cdot)+\alpha V_{2}(\cdot), \quad n \rightarrow \infty
\end{align*}
$$

in $\mathbb{M}\left([0, \infty]^{2} \backslash\{\mathbf{0}\}\right)$. We justify the approximation in (5.11) and the convergence result is summarized in the following theorem.

Theorem 5.4. Suppose that $\left\{k_{n}\right\}$ is an intermediate sequence satisfying

$$
\begin{equation*}
\liminf _{n \rightarrow \infty} k_{n} /(n \log n)^{1 / 2}>0 \quad \text { and } \quad k_{n} / n \rightarrow 0 \quad \text { as } \quad n \rightarrow \infty \tag{5.12}
\end{equation*}
$$

and recall $a=c_{\text {out }} / c_{i n}$. Then we have
in $M_{+}\left([0, \infty]^{2} \backslash\{\mathbf{0}\}\right)$, where $V_{1}$ and $V_{2}$ concentrate on $(0, \infty)^{2}$ with Lebesgue densities

$$
\begin{equation*}
f_{1}(x, y)=\frac{x^{\delta_{i n}} y^{\delta_{o u t}-1}}{c_{\text {in }} \Gamma\left(1+\delta_{\text {in }}\right) \Gamma\left(\delta_{\text {out }}\right)} \iint^{\infty} z^{-\left(2+1 / c_{\text {in }}+\delta_{\text {in }}+a \delta_{\text {out }}\right)} e^{-x / z+y / z^{a}} \mathrm{~d} z \tag{5.14}
\end{equation*}
$$

and

$$
\begin{equation*}
f_{2}(x, y)=\frac{x^{\delta_{\text {in }}-1} y^{\delta_{\text {out }}}}{c_{\text {in }} \Gamma\left(\delta_{\text {in }}\right) \Gamma\left(1+\delta_{\text {out }}\right)} \iint^{\infty} z^{-\left(1+a+1 / c_{\text {in }}+\delta_{\text {in }}+a \delta_{\text {out }}\right)} e^{-x / z+y / z^{a}} \mathrm{~d} z \tag{5.15}
\end{equation*}
$$

respectively.
Proof. Proving (5.13) requires using concentration results for degree counts $N_{i, j}(n)$ which compare counts with expected counts; these are collected in Section 7. In this section we show for $x, y>0$,

$$
\begin{align*}
& \frac{1}{k_{n}} \mathbf{E}\left(N_{>\left(\frac{n}{k_{n}}\right)^{c_{\text {in }}} x,>\left(\frac{n}{k_{n}}\right)^{c_{\text {out }}} y}(n)\right)-\frac{n}{k_{n}} p_{>\left(\frac{n}{k_{n}}\right)^{c_{\text {in }}} x,>\left(\frac{n}{k_{n}}\right)^{c_{\text {out }}} y \xrightarrow{P} 0, ~} \quad 0,  \tag{5.16a}\\
& \frac{1}{k_{n}} \mathbf{E}\left(N_{>\left(\frac{n}{k_{n}}\right)^{\mathrm{in}} x}^{\mathrm{cin}_{x}}(n)\right)\left(-\frac{n}{k_{n}} p_{>\left(\frac{n}{k_{n}}\right)^{\mathrm{in}} x} \quad \stackrel{P}{c_{\mathrm{in}}} 0,\right.  \tag{5.16b}\\
& \frac{1}{k_{n}} \mathbf{E}\left(N_{>\left(\frac{n}{k_{n}}\right)^{\text {out }} y}^{c_{\text {out }}}(n)\right)\left(-\frac{n}{k_{n}} p_{>\left(\frac{n}{k_{n}}\right)^{\text {out }} y}^{c^{\text {out }} y} \xrightarrow{P} 0 .\right. \tag{5.16c}
\end{align*}
$$

We give a proof for (5.16a) and (5.16b) and (5.16c) follows from a similar argument.

Adopting the notation from the proof of Theorem 5.1, and using (5.8) we have

$$
\begin{aligned}
& \frac{1}{k_{n}} \mathbf{E}\left(N_{>\left(\frac{n}{k_{n}}\right)^{c_{\text {in }}} x,>\left(\frac{n}{k_{n}}\right)^{c_{\text {out }}} y}(n)\right)-\frac{n}{k_{n}} p_{>\left(\frac{n}{k_{n}}\right)^{c_{\text {in }}} x,>\left(\frac{n}{k_{n}}\right)^{c_{\text {out }}} y} \\
& =\frac{n}{k_{n}} \frac{1}{n} \sum_{v \in[n]} \not\left(\mathbb{P}\left(\frac{D_{v}^{\text {in }}(n)}{\left(n / k_{n}\right)^{c_{\text {in }}}}>x, \frac{D_{v}^{\text {out }}(n)}{\left(n / k_{n}\right)^{c_{\text {out }}}}>y\right)-\frac{n}{k_{n}} \mathbf{P}\left[\frac{\mathcal{I}}{\left(n / k_{n}\right)^{c_{\text {in }}}}>x, \frac{\mathcal{O}}{\left(n / k_{n}\right)^{c_{\text {out }}}}>y\right]\right.
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{n}{k_{n}} \mathbf{P}\left[\frac{\mathcal{I}}{\left(n / k_{n}\right)^{c_{\text {in }}}}>x, \frac{\mathcal{O}}{\left(n / k_{n}\right)^{c_{\text {out }}}}>y\right]( \\
& +\frac{1}{k_{n}} \mathbf{P}\left(\frac{D_{1}^{\text {in }}(n)}{\left(n / k_{n}\right)^{c_{\text {in }}}}>x, \frac{D_{1}^{\text {out }}(n)}{\left(n / k_{n}\right)^{c_{\text {out }}}}>y\right)(
\end{aligned}
$$

$$
\begin{aligned}
& \text { as } n \rightarrow \infty \text {. }
\end{aligned}
$$

Combining concentration results in (7.1), (7.5) and (7.6) with (5.16) implies that for any intermediate sequence $\left\{k_{n}\right\}$ satisfying (5.12) and $x, y>0$, as $n \rightarrow \infty$,

$$
\begin{gather*}
\frac{1}{k_{n}} N_{>\left(\frac{n}{k_{n}}\right)^{c_{\text {in }}} x,>\left(\frac{n}{k_{n}}\right)^{c_{\text {out }}} y}(n)-n p_{>\left(\frac{n}{k_{n}}\right)^{c_{\text {in }}} x,>\left(\frac{n}{k_{n}}\right)^{c_{\text {out }}} y \xrightarrow{P} 0,}^{\quad \frac{1}{k_{n}} N_{>\left(\frac{n}{k_{n}}\right)^{c_{\text {in }}} x}^{\text {in }}(n)-n p_{>\left(\frac{n}{k_{n}}\right)^{\text {in }}}^{c_{\text {in }}} \xrightarrow{P} 0,}  \tag{5.17a}\\
\frac{1}{k_{n}} N_{>\left(\frac{n}{k_{n}}\right)^{\text {out }}{ }^{c_{\text {out }}}(n)-n p_{>\left(\frac{n}{k_{n}}\right)^{\text {out }} y}^{c_{\text {out }}} \xrightarrow{P} 0 .} . \tag{5.17b}
\end{gather*}
$$

Define the vague metric $\rho(\cdot, \cdot)$ on $M_{+}\left([0, \infty]^{2} \backslash\{\mathbf{0}\}\right.$ ) (cf. [18, Chapter 3.3]) as follows. There exists some sequence of continuous functions on $[0, \infty]^{2} \backslash\{\mathbf{0}\}$ with compact supports, $f_{i}$ : $[0, \infty]^{2} \backslash\{\mathbf{0}\} \mapsto \mathbb{R}_{+}, i \geq 1$, and for $\mu_{1}, \mu_{2} \in M_{+}\left([0, \infty]^{2} \backslash\{\mathbf{0}\}\right)$,

$$
\rho\left(\mu_{1}, \mu_{2}\right)=\sum_{i=1}^{\infty} \frac{\left|\mu_{1}\left(f_{i}\right)-\mu_{2}\left(f_{i}\right)\right| \wedge 1}{2^{i}}
$$

where $\mu_{j}\left(f_{i}\right):=\int_{\left[\{, \infty]^{2} \backslash\{0\}\right.} f_{i}(x) \mu_{j}(\mathrm{~d} x), j=1,2, i \geq 1$. Then results in (5.17) imply: as (5.18)

$$
\rho\left(\frac{1}{k_{n}} \sum_{v \in[n]}^{n} \epsilon_{\left(\varphi_{v}^{\text {in }}(n) /\left(n / k_{n}\right)^{c_{\text {in }}}, D_{v}^{\text {out }}(n) /\left(n / k_{n}\right)^{c_{\text {out }}}\right)}, \frac{n}{k_{n}} \mathbf{P}\left[\left(\left(\frac{\mathcal{I}}{\left.n / k_{n}\right)^{c_{\mathrm{in}}}}, \frac{\mathcal{O}}{\left(n / k_{n}\right)^{c_{\text {out }}}}\right) \in \cdot\right]\right)\left({ }_{P}^{P} 0 .\right.\right.
$$

Then (5.13) follows from combining (5.18) and the vague convergence in (5.10), with (5.14) and (5.15) being specified in [20, Theorem 2].

## 6. Consistency of the Hill Estimator

In practice, the growth rates of in- and out-degrees are often estimated by Hill estimators as defined in (1.4). However, despite its wide use, there is no theoretical justification for such estimates and the consistency has been proved only for a simple undirected preferential attachment model in [27]. We now turn to (1.7) and (1.8) as preparations for considering consistency of the Hill estimator.

Proposition 6.1. Suppose that $\left\{k_{n}\right\}$ is some intermediate sequence satisfying (5.12). Define

$$
\begin{aligned}
& b_{1}(t)=\left[c_{\text {in }} \frac{\Gamma\left(1+\delta_{\text {in }}+c_{\text {in }}^{-1}\right)}{\Gamma\left(1+\delta_{\text {in }}\right)}\left(\frac{\alpha \delta_{\text {in }}}{1+c_{\text {in }} \delta_{\text {in }}}+\frac{\gamma}{c_{\text {in }}}\right)\right]^{c_{\text {in }}} t^{c_{\text {in }}} \\
& b_{2}(t)=\left[\left\{_{\text {out }} \frac{\Gamma\left(1+\delta_{\text {out }}+c_{\text {out }}^{-1}\right)}{\Gamma\left(1+\delta_{\text {out }}\right)}\left(\left(\frac{\gamma \delta_{\text {out }}}{c_{\text {out }} \delta_{\text {out }}}+\frac{\alpha}{c_{\text {out }}}\right)\right]^{c_{\text {out }}} t^{c_{\text {out }}}\right.\right.
\end{aligned}
$$

then

$$
\begin{array}{ll}
\frac{1}{k_{n}} \sum_{v \in[n]} \epsilon_{D_{v}^{i n}(n) / b_{1}\left(n / k_{n}\right)} \Rightarrow \nu_{c_{\text {in }}^{-1}}, & \text { in } M_{+}((0, \infty]), \\
\frac{1}{k_{n}} \sum_{v \in[n]} \epsilon_{D_{v}^{o u t}(n) / b_{2}\left(n / k_{n}\right)} \Rightarrow \nu_{c_{\text {out }}^{-1}}, & \text { in } M_{+}((0, \infty]) . \tag{6.2}
\end{array}
$$

Proof. Marginalizing the results in (5.13) gives

$$
\begin{aligned}
& \frac{1}{k_{n}} \sum_{v \in[n]} \xi_{\frac{\operatorname{in}^{\mathrm{in}(n)}}{\left(n / k_{n}\right)^{\mathrm{in}}}} \Rightarrow c_{\mathrm{in}} \frac{\Gamma\left(1+\delta_{\mathrm{in}}+c_{\mathrm{in}}^{-1}\right)}{\Gamma\left(1+\delta_{\mathrm{in} n}\right)}\left(\frac{\alpha \delta_{\mathrm{in}}}{1+c_{\mathrm{in}} \delta_{\mathrm{in}}}+\frac{\gamma}{c_{\mathrm{in}}}\right)\left(\chi_{c_{\mathrm{in}}^{-1}}, \quad \text { in } M_{+}((0, \infty]),\right. \\
& \frac{1}{k_{n}} \sum_{v \in[n]} \epsilon_{\frac{D^{\text {out }}(n)}{\left(n / k_{n}\right)^{\text {out }}}} \Rightarrow c_{\text {out }} \frac{\Gamma\left(1+\delta_{\text {out }}+c_{\text {out }}^{-1}\right)}{\Gamma\left(1+\delta_{\text {out }}\right)}\left(\frac{\gamma \delta_{\text {out }}}{1+c_{\text {out }} \delta_{\text {out }}}+\frac{\alpha}{c_{\text {out }}}\right)\left(c_{c_{\text {out }}^{-1}}, \quad \text { in } M_{+}((0, \infty])\right. \text {. }
\end{aligned}
$$

Scaling both sides by the constant appearing in the limit measure gives (6.1) and (6.2).
With Proposition 6.1 available, we now prove the consistency of Hill estimators for in- and out-degrees.

Theorem 6.2. Let

$$
\begin{aligned}
& D_{(1)}^{i n}(n) \geq D_{(2)}^{i n}(n) \geq \cdots \geq D_{(n)}^{i n}(n) \\
& D_{(1)}^{\text {out }}(n) \geq D_{(2)}^{\text {out }}(n) \geq \cdots \geq D_{(n)}^{\text {out }}(n)
\end{aligned}
$$

be order statistics for in- and out-degrees $\left\{D_{v}^{i n}(n)\right\}_{v \in[n]},\left\{D_{v}^{\text {out }}(n)\right\}_{v \in[n]}$, respectively. Define the Hill estimators for $\left\{D_{v}^{i n}(n)\right\}_{v \in[n]}$ and $\left\{D_{v}^{\text {out }}(n)\right\}_{v \in[n]}$ as

$$
H_{k, n}^{i n}:=\frac{1}{k} \sum_{i=1}^{k} \log \frac{D_{(i)}^{i n}(n)}{D_{(k+1)}^{i n}(n)}, \quad H_{k, n}^{\text {out }}:=\frac{1}{k} \sum_{i=1}^{k}\left(\operatorname{og} \frac{D_{(i)}^{\text {out }}(n)}{D_{(k+1)}^{\text {out }}(n)} .\right.
$$

Then for some intermediate sequence $\left\{k_{n}\right\}$ satisfying (5.12), we have as $n \rightarrow \infty$,

$$
\begin{equation*}
H_{k_{n}, n}^{i n} \xrightarrow{P} c_{\text {in }}, \quad \text { and } \quad H_{k_{n}, n}^{\text {out }} \xrightarrow{P} c_{\text {out }} . \tag{6.3}
\end{equation*}
$$

Proof. From (6.1) and (6.2), we conclude by inversion and [18, Proposition 3.2] that in $D(0, \infty]$

$$
\frac{D_{\left(\left[k_{n} t\right]\right)}^{\text {in }}(n)}{b_{1}\left(n / k_{n}\right)} \xrightarrow{P} t^{-c_{\text {in }}} \quad \text { and } \quad \frac{D_{\left(\left[k_{n} t\right]\right)}^{\text {out }}(n)}{b_{2}\left(n / k_{n}\right)} \xrightarrow{P} t^{-c_{\text {out }}} .
$$

Therefore,

$$
\begin{align*}
& \left(\frac{1}{\not / f_{n}} \sum_{v \in[n]} \xi_{D_{v}^{\text {in }}(n) / b_{1}\left(n / k_{n}\right)}, \frac{D_{\left(k_{n}\right)}^{\text {in }}(n)}{b_{1}\left(n / k_{n}\right)}\right)\left(\nu_{c_{\text {in }}^{-1}}, 1\right)\left(\text { in } M_{+}((0, \infty]) \times(0, \infty),\right.  \tag{6.4}\\
& \left(\frac{1}{\nmid k_{n}} \sum_{v \in[n]} \epsilon_{D_{v}^{\text {out }}(n) / b_{2}\left(n / k_{n}\right)}, \frac{D_{\left(k_{n}\right)}^{\text {out }}(n)}{b_{2}\left(n / k_{n}\right)}\right) \Leftrightarrow\left(\nu_{c_{\text {out }}^{-1}}, 1\right)\left(\text { in } M_{+}((0, \infty]) \times(0, \infty) .\right.
\end{align*}
$$

$$
S: M_{+}((0, \infty]) \times(0, \infty) \mapsto M_{+}((0, \infty])
$$

by

$$
S(\nu, c)(A)=\nu(c A)
$$

By the proof in [18, Theorem 4.2], the mapping $S$ is continuous at $\left(\nu_{c_{i}^{-1}}, 1\right), i=1,2$. Therefore, applying the continuous mapping $S$ to the joint weak convergence in (6.4) and (6.5) gives

$$
\begin{array}{ll}
\frac{1}{k_{n}} \sum_{v \in[n]} \epsilon_{D_{v}^{\text {in }}(n) / D_{\left(k_{n}\right)}^{\mathrm{in}}(n)} \Rightarrow \nu_{c_{\text {in }}^{-1}}, & \text { in } M_{+}((0, \infty]), \\
\frac{1}{k_{n}} \sum_{v \in[n]} \epsilon_{D_{v}^{\text {out }}(n) / D_{\left(k_{n}\right)}^{\text {out }}(n)} \Rightarrow \nu_{c_{\text {out }}^{-1}}, & \text { in } M_{+}((0, \infty]) .
\end{array}
$$

Then the rest of the proof is similar to arguments in the proof of [27, Theorem 11]. Here we only include proofs for the consistency $H_{k_{n}, n}^{\text {in }}$ and that for $H_{k_{n}, n}^{\text {out }}$ follows from the same argument. Define $\hat{\nu}_{n}^{\operatorname{in}}(\cdot):=\frac{1}{k_{n}} \sum_{y}\left(\in[n]{ }_{D_{v}^{\text {in }}(n) / D_{\left(k_{n}\right)}^{\text {in }}(n)}(\cdot)\right.$. First observe

$$
H_{k_{n}, n}^{\mathrm{in}}=\int_{1}^{\infty}\left(\hat{\nu}_{n}^{\mathrm{in}}(y, \infty] \frac{\mathrm{d} y}{y} .\right.
$$

Then fix $M>0$ large and define a mapping $f \mapsto \int_{1}^{M} f(y) \frac{\mathrm{d} y}{y}$ from $D(0, \infty] \mapsto \mathbb{R}_{+}$. This map is a.s. continuous so

$$
\int_{1}^{M} \hat{\nu}_{n}^{\text {in }}(y, \infty] \frac{\mathrm{d} y}{y} \xrightarrow{P} \int_{1}^{M} \nu_{c_{\mathrm{in}}^{-1}}(y, \infty] \frac{\mathrm{d} y}{y},
$$

and it remains to show by the second converging together theorem (cf. [18, Theorem 3.5]) that

$$
\begin{equation*}
\lim _{M \rightarrow \infty} \limsup _{n \rightarrow \infty} \mathbf{P}\left(\iint_{N}^{\infty} \hat{\nu}_{n}^{\text {in }}(y, \infty] \frac{\mathrm{d} y}{y}>\varepsilon\right)=0 . \tag{6.6}
\end{equation*}
$$

The probability in (6.6) is

$$
\begin{aligned}
& \mathbf{P}\left(\int\left({ }_{x}^{\infty} \hat{\nu}_{n}^{\text {in }}(y, \infty] \frac{\mathrm{d} y}{y}>\varepsilon\right) \leq \mathbf{P} \int\left({ }_{(1}^{\infty} \hat{\nu}_{n}^{\text {in }}(y, \infty] \frac{\mathrm{d} y}{y}>\varepsilon, \frac{D_{\left(k_{n}\right)}^{\text {in }}(n)}{b_{1}\left(n / k_{n}\right)}-1<\eta\right)\right. \\
& \left.+\mathbf{P} \iint_{\{ }^{\infty} \hat{\nu}_{n}^{\mathrm{in}}(y, \infty] \frac{\mathrm{d} y}{y}>\varepsilon, \frac{D_{\left(k_{n}\right)}^{\mathrm{in}}(n)}{b_{1}\left(n / k_{n}\right)}-1 \geq \eta\right)( \\
& \leq \mathbf{P} \int\left({ }_{(1}^{\infty} \frac{1}{k_{n}} \sum_{i=1}^{n} \epsilon_{D_{i}^{\text {in }}(n) / b_{1}\left(n / k_{n}\right)}((1-\eta) y, \infty] \frac{\mathrm{d} y}{y}>\varepsilon\right) \\
& \left.+\mathbf{P} \quad \frac{D_{\left(k_{n}\right)}^{\mathrm{in}}(n)}{b_{1}\left(n / k_{n}\right)}-1 \geq \eta\right)=: A+B .
\end{aligned}
$$

By (6.4), $B \rightarrow 0$ as $n \rightarrow \infty$, and using the Markov inequality, $A$ is bounded by

$$
\begin{aligned}
& \left.\frac{1}{\varepsilon} \mathbf{E} \iint_{(K}^{\infty} \frac{1}{k_{n}} \sum_{v=1}^{n} \epsilon_{D_{v}^{\text {in }}(n) / b_{1}\left(n / k_{n}\right)}((1-\eta) y, \infty] \frac{\mathrm{d} y}{y}\right) \\
= & \left.\frac{1}{\varepsilon} \mathbf{E} \iint_{((1-\eta)}^{\infty} \frac{1}{k_{n}} \sum_{v=1}^{n} \oint_{D_{v}^{\text {in }(n) / b_{1}\left(n / k_{n}\right)}}(y, \infty] \frac{\mathrm{d} y}{y}\right) \leq \frac{1}{\varepsilon} \int_{M(1-\eta)}^{\infty} \frac{1}{k_{n}} \mathbf{E}\left(N_{>\left[b_{1}\left(n / k_{n}\right) y\right]}^{\mathrm{in}}(n)\right) \frac{\mathrm{d} y}{y} .
\end{aligned}
$$

Using Stirling's formula, (5.17b) gives that for $y>0$,

$$
\begin{equation*}
\frac{1}{k_{n}} \mathbf{E}\left(N_{>\left[b_{1}\left(n / k_{n}\right) y\right]}^{\mathrm{in}}(n)\right) \nrightarrow y^{-c_{\mathrm{in}}^{-1}} \tag{6.7}
\end{equation*}
$$

Let $U(t):=\mathbf{E}\left(N_{>t}^{\mathrm{in}}(n)\right)$ and (6.7) becomes: for $y>0$,

$$
\frac{1}{k_{n}} U\left(b_{1}\left(n / k_{n}\right) y\right) \rightarrow y^{-c_{\mathrm{in}}^{-1}}, \quad \text { as } n \rightarrow \infty .
$$

Since $U(\cdot)$ is a non-increasing function, $U \in R V_{-c_{\mathrm{in}}^{-1}}$ by [18, Proposition 2.3(ii)]. Therefore, Karamata's theorem gives

$$
A \leq \frac{1}{\varepsilon} \int_{\langle((1-\eta)}^{\infty} \frac{1}{k_{n}} \mathbf{E}\left(\not X_{>\left[b_{1}\left(n / k_{n}\right) y\right]}^{\mathrm{in}}(n)\right) \frac{\mathrm{d} y}{y} \sim C(\delta, \eta) M^{-c_{\text {in }}^{-1}},
$$

with some positive constant $C(\delta, \eta)>0$. Also, $M^{-c_{\mathrm{in}}^{-1}} \rightarrow 0$ as $M \rightarrow \infty$, and (6.6) follows.

## 7. Concentration of Degree counts

In this section, we collect concentration results for the degree counts that are useful in the proofs in Theorem 5.4.

Lemma 7.1. Define $N_{>i,>j}(n):=\sum_{q} \in\left[\in[n] \mathbb{1}_{\left\{D_{v}^{i n}(n)>i, D_{v}^{\text {out }(n)>i\}}\right.}\right.$. Then for $\delta_{\text {in }}>0$, there exists a constant $C>6$ such that as $n \rightarrow \infty$

$$
\begin{equation*}
\boldsymbol{P}\left(\max _{i, j}\left|N_{>i,>j}(n)-\boldsymbol{E}\left(N_{>i,>j}(n)\right)\right| \geq C(1+\sqrt{\eta \log n})\right\}=o(1) \tag{7.1}
\end{equation*}
$$

Proof. The proof of (7.1) follows from a similar argument as in the proof of [22, Proposition 8.4]. We include it here to make it self-contained. Define a martingale

$$
M_{m}:=\mathbf{E}\left(X_{>i,>j}(n) \mid G(m)\right)=\sum_{v \in[n]} \mathbf{P}\left(\not \Phi_{v}^{\text {in }}(n)>i, D_{v}^{\text {out }}(n)>j \mid G(m)\right)(m \leq n .
$$

For $m \geq 2$, we define a new graph $G^{\prime}(s)$ by $G^{\prime}(s)=G(s)$ for $s \leq m-1$, while $s \mapsto G^{\prime}(m)$ evolves independently of $\{G(s): s \geq m-1\}$, following the preferential attachment rule given in Section 2.1. Denote the in- and out-degrees of the node $v$ in $G^{\prime}(n)$ by $\left(D^{\text {in }}\right)_{v}^{\prime}(n),\left(D^{\text {out }}\right)_{v}^{\prime}(n)$, we then have

$$
\begin{equation*}
M_{m-1}=\sum_{v \in[n]} \mathbf{P}\left(\left({\left(D^{\mathrm{in}}\right)}_{v}^{\prime}(n)>i,\left(D^{\text {out }}\right)_{v}^{\prime}(n)>j \mid G(m-1)\right)\right. \tag{7.2}
\end{equation*}
$$

Since the evolution of $s \mapsto G^{\prime}(s)$ is independent of that of $\{G(s): s \geq m-1\}$ for $s \geq m-1$, it makes no difference whether we condition on $G(m-1)$ or $G(m)$ in (7.2). Hence, we have

$$
\begin{align*}
& M_{m}-M_{m-1}  \tag{7.3}\\
& \quad=\sum_{v \in[n]}\left[\mathbf{P}\left(\not \prod_{v}^{\text {in }}(n)>i, D_{v}^{\text {out }}(n)>j G(m)\right)-\mathbf{P}\left(\left(\left(D^{\text {in }}\right)_{v}^{\prime}(n)>i,\left(D^{\text {out }}\right)_{v}^{\prime}(n)>j G(m)\right)\right] .( \right.
\end{align*}
$$

Since the evolution of $n \mapsto\left(D_{v}^{\text {in }}(n), D_{v}^{\text {out }}(n)\right)$ for $n \geq m$ only depends on $\left(D_{v}^{\text {in }}(m), D_{v}^{\text {out }}(m)\right)$, then

$$
\begin{aligned}
& \quad \mathbf{P}\left(\not D_{v}^{\text {in }}(n)>i, D_{v}^{\text {out }}(n)>j G(m)\right)=\mathbf{P}\left(\not D_{v}^{\text {in }}(n)>i, D_{v}^{\text {out }}(n)>j\left(D_{v}^{\text {in }}(m), D_{v}^{\text {out }}(m)\right)\right),( \\
& \quad \mathbf{P}\left(\left(D^{\text {in }}\right)_{v}^{\prime}(n)>i,\left(D^{\text {out }}\right)_{v}^{\prime}(n)>j G(m)\right) \\
& \quad=\mathbf{E}\left\{\mathbf { P } \left[\left(\left(D^{\text {in }}\right)_{v}^{\prime}(n)>i,\left(D^{\text {out }}\right)_{v}^{\prime}(n)>j\left(\left(D^{\text {in }}\right)_{v}^{\prime}(m),\left(D^{\text {out }}\right)_{v}^{\prime}(m)\right)\right](G(m)\} .\right.\right. \\
& \text { Then }(7.3) \text { becomes }
\end{aligned}
$$

$$
\begin{align*}
& M_{m}-M_{m-1}  \tag{7.4}\\
& =\sum_{v \in[n]} \mathbf{E}\left\{\mathbf{P}\left[\not \phi_{v}^{\text {in }}(n)>i, D_{v}^{\text {out }}(n)>j\left(D_{v}^{\text {in }}(m), D_{v}^{\text {out }}(m)\right)\right]( \right. \\
& \left.\left.-\mathbf{P}\left[\left(D^{\text {in }}\right)_{v}^{\prime}(n)>i,\left(D^{\text {out }}\right)_{v}^{\prime}(n)>j\left(\left(D^{\text {in }}\right)_{v}^{\prime}(m),\left(D^{\text {out }}\right)_{v}^{\prime}(m)\right)\right]\right\}(m)\right\} . \\
& \text { ortant to note that }
\end{align*}
$$

$$
\begin{array}{r}
\mathbf{P}\left[\not \ddot{P}_{v}^{\text {in }}(n)>i, D_{v}^{\text {out }}(n)>j\left(D_{v}^{\text {in }}(m), D_{v}^{\text {out }}(m)\right)\right] \\
\quad=\mathbf{P}\left[\left(\rho^{\text {in }}\right)_{v}^{\prime}(n)>i,\left(D^{\text {out }}\right)_{v}^{\prime}(n)>\left(j\left(\left(D^{\text {in }}\right)_{v}^{\prime}(m),\left(D^{\text {out }}\right)_{v}^{\prime}(m)\right)\right],\right.
\end{array}
$$

as long as $\left(D_{v}^{\text {in }}(m), D_{v}^{\text {out }}(m) \ell=\left(\left(D^{\text {in }}\right)_{v}^{\prime}(m),\left(D^{\text {out }}\right)_{v}^{\prime}(m)\right)\right.$, (because the two graphs are constructed based on the same preferential attachment rule. Thus,

$$
\begin{aligned}
& \mathbf{P}\left[\Phi_{v}^{\text {in }}(n)>i, D_{v}^{\text {out }}(n)>j\left(D_{v}^{\text {in }}(m), D_{v}^{\text {out }}(m)\right)\right] \\
& \quad-\mathbf{P}\left[\left(D^{\text {in }}\right)_{v}^{\prime}(n)>i,\left(D^{\text {out }}\right)_{v}^{\prime}(n)>j\left(\left(D^{\text {in }}\right)_{v}^{\prime}(m),\left(D^{\text {out }}\right)_{v}^{\prime}(m)\right)\right](
\end{aligned}
$$

$$
\begin{aligned}
& \leq 1 \quad\left\{\left(D_{v}^{\text {in }}(m), D_{v}^{\text {out }}(m)\right) \neq\left(\left(\phi^{\text {in }}\right)_{v}^{\prime}(m),\left(D^{\text {out }}\right)_{v}^{\prime}(m)\right)\right\}^{.} \\
& \text {(7.4) is bounded by: }
\end{aligned}
$$

So we conclude that (7.4) is bounded by:

$$
\begin{aligned}
& \left|M_{m}-M_{m-1}\right| \\
& \leq \sum_{v \in[n]} \mathbf{E}\left\{\mathbf{P}\left[\phi_{v}^{\text {in }}(n)>i, D_{v}^{\text {out }}(n)>j\left(D_{v}^{\text {in }}(m), D_{v}^{\text {out }}(m)\right)\right]\right\} \\
& -\mathbf{P}\left[\left(\left(D^{\text {in }}\right)_{v}^{\prime}(n)>i,\left(D^{\text {out }}\right)_{v}^{\prime}(n)>j\left(\left(D^{\text {in }}\right)_{v}^{\prime}(m),\left(D^{\text {out }}\right)_{v}^{\prime}(m)\right)\right](G(m)\}\right. \\
& \leq \sum_{v \in[n]} \mathbb{E}^{1} \begin{array}{l}
\left\{\left(D_{v}^{\text {in }}(m), D_{v}^{\text {out }}(m)\right) \neq\left(\left(D^{\text {in }}\right)_{v}^{\prime}(m),\left(D^{\text {out }}\right)_{v}^{\prime}(m)\right)\right\} \\
G(m))
\end{array} \\
& =\mathbf{E}\left(\sum_{\langle[n]}\left\{\left(\left\{^{\text {in }}(m), D_{v}^{\text {out }}(m)\right) \neq\left(\left(\phi^{\text {in }}\right)_{v}^{\prime}(m),\left(D^{\text {out }}\right)_{v}^{\prime}(m)\right)\right\}^{G(m)}\right)\right\}
\end{aligned}
$$

Note that $\left(D_{v}^{\text {in }}(m-1), D_{v}^{\text {out }}(m-1)\right) \neq\left(\left(D^{\text {in }}\right)_{v}^{\prime}(m-1),\left(D^{\text {out }}\right)_{v}^{\prime}(m-1)\right)$ for all $1 \leq v \leq m-1$ by construction, and since changing an edge will change the in- and out-degrees for at most
3 nodes, then 3 nodes, then

$$
\left|M_{m}-M_{m-1}\right| \leq 3
$$

Next, we use the Azuma-Hoeffding inequality to prove (7.1). Since $N_{>i,>j}(n)=0$ for $i, j>n$, then

$$
\begin{aligned}
& \mathbf{P}\left(\max _{i, j}\left|N_{>i,>j}(n)-\mathbf{E}\left(N_{>i,>j}(n)\right)\right| \geq C \sqrt{n \log n}\right) \\
& \quad \leq \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \not p\left(\left|N_{>i,>j}(n)-\mathbf{E}\left(N_{>i,>j}(n)\right)\right| \geq C \sqrt{n \log n}\right) \\
& \quad \leq n^{2} \cdot 2 \exp \left\{-\frac{C^{2} \log n}{2 \cdot 3^{2}}\right\}\left(=2 n^{-\left(C^{2} / 18-2\right)} .\right.
\end{aligned}
$$

Therefore, (7.1) follows from taking $C>6$.
Results in Lemma 7.2 also follows from the argument in [22, Proposition 8.4] Since the details of this proof machinery has been given in the proof of Lemma 7.1, they are omitted here.

Lemma 7.2. For $\delta_{\text {in }}, \delta_{\text {out }}>0$, there exist constants $C_{\text {in }}, C_{\text {out }}>3 \sqrt{2}$, such that as $n \rightarrow \infty$,

$$
\begin{equation*}
\boldsymbol{P}\left(\max _{i \geq 0} N_{>i}^{i n}(n)-\boldsymbol{E}\left(N_{>i}^{i n}(n)\right) \geq C_{i n}(1+\sqrt{n \log n})\right)=o(1) \tag{7.5}
\end{equation*}
$$

and

$$
\begin{equation*}
\boldsymbol{P}\left(\max _{j \geq 0} N_{>j}^{\text {out }}(n)-\boldsymbol{E}\left(N_{>i}^{o u t}(n)\right) \geq C_{o u t}(1+\sqrt{n \log n})\right)=o(1) . \tag{7.6}
\end{equation*}
$$

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