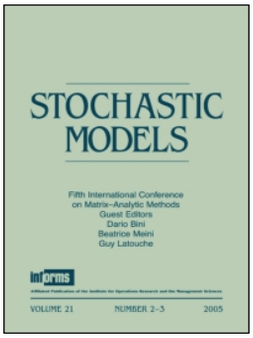

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Asymptotic normality of in- and out-degree counts in a preferential attachment model

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ABSTRACT
Preferential attachment in a directed scale-free graph is an often used paradigm for modeling the evolution of social networks. Social network data is usually given in a format allowing recovery of the number of nodes with in-degree i and out-degree j. Assuming a model with preferential attachment, formal statistical procedures for estimation can be based on such data summaries. Anticipating the statistical need for such node-based methods, we prove asymptotic normality of the node counts. Our approach is based on a martingale construction and a martingale central limit theorem.

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1. Introduction

Preferential attachment for both undirected and directed scale-free graphs has been introduced as a model for the growth of social networks (cf. Refs.[2,3,9,11]) and has been suggested for contexts such as the web graph, citation graph, co-author graph, etc. Data examples and analyses are given at http://konect.uni-koblenz.de/networks/ and https://snap.stanford.edu/data/. Attention is focused on the directed case where nodes typically have at least two characteristics, namely in- and out-degree. More characteristics are possible, as for example, in recommender networks such as *slashdot* (https://slashdot.org) but we confine our attention to in- and out-degree.

In preferential attachment networks, nodes with large in-degrees tend to attract more followers from new nodes, whereas existing nodes with large out-degrees tend to become followers of new nodes. Although this model is relatively naive and not guaranteed to fit real data, preferential attachment captures features of a real social networks and is a useful paradigm on which to test statistical and computational methodology.

Social network data is often formatted as lines of text giving a time and an ordered pair of nodes from which a directed edge is inferred. For example, this is one of the preferred formats of the R-package *igraph* [5] and from this format, one can compute the number of nodes with given in- and out-degree. Let $N_n(i, j)$ be the number of

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nodes with in-degree i and out-degree j in a directed preferential attachment model (cf. Ref.[3]) at the nth step of the growth of the network. Under simple preferential attachment assumptions, Bollobás et al.[3] showed that $N_n(i, j)/n \to p_{ij}$ for fixed i and j, and provided an explicit form of (p_{ij}). Under the same assumptions, we also know that the limiting degree sequence (p_{ij}) has both marginally and jointly regularly varying tails (cf. Refs.[3,13,15,17]).

The goal of this paper is to examine the asymptotic normality of $N_n(i, j)$ with the idea that this asymptotic normality can justify statistical estimation methods in practice. Since data from social networks are node based, which in nature are nothing like those collected from independent repeated sampling, the natural tool is the martingale central limit theorem. We will show asymptotic normality of $\sqrt{n}(N_n(i, j)/n - p_{ij})$ for fixed (i, j) as well as jointly over (i, j). This will imply that the empirical estimator $N_n(i, j)/n$ is consistent and asymptotically normal. We defer an exploration of more formal statistical inference for our preferential attachment model that relies on node-based data and asymptotic normality.

The directed preferential attachment model that we study is outlined in Section 2 and our main results on normality are summarized in Section 3. Proofs are collected in Section 4.

2. The preferential attachment model

We somewhat simplify the model used in Refs.[3,13,15]. At each step of the construction, a node is added; we exclude the possibility of adding only a new edge between existing nodes. The model evolves according to the following dynamics. Choose strictly positive parameters α, γ, λ and μ such that $\alpha + \gamma = 1$, and we assume in addition that $\alpha, \gamma < 1$ to avoid trivial cases.

We initiate the algorithm with a simple case: A graph G_1 with one single node (labeled 1) with a self-loop so that both its in and out degrees are 1, denoted by $D_1(1) = (1, 1)$. At stage n, we have a directed random graph $G_n = (V_n, E_n)$. If a node v is from V_n, use $D_{\text{in}}(v)$ and $D_{\text{out}}(v)$ to denote its in- and out-degree, respectively, (dependence on n is suppressed) and write $D_n(v) = (D_{\text{in}}(v), D_{\text{out}}(v))$. Then, G_{n+1} is obtained from G_n as follows.

(i) With probability α a new node w is born and we add an edge leading from w to an existing node $v \in V_n$. The existing node v is chosen with probability according to its in-degree:

$$\mathbf{P}(v \in V_n \text{ is chosen}) = \frac{D_{\text{in}}(v) + \lambda}{(1 + \lambda)n}. \tag{2.1}$$

(ii) With probability γ a new node w is born and we add an edge leading from an existing node $v \in V_n$ to w. The existing node v is chosen with probability according to its out-degree:

$$\mathbf{P}(v \in V_n \text{ is chosen}) = \frac{D_{\text{out}}(v) + \mu}{(1 + \mu)n}. \tag{2.2}$$

The construction makes G_n a directed graph with n nodes (i.e., $V_n = \{1, 2, \ldots, n\}$) and $n-1$ edges; the self-loop in G_1 is not counted as an edge. Note that

$$\sum_{v \in V_n} D_{\text{in}}(v) = \sum_{v \in V_n} D_{\text{out}}(v) = n,$$

so the attachment probabilities in (2.1) and (2.2) add to 1.

3. Statement of results

For $i, j \geq 0$, let $N_n(i, j)$ denote the number of nodes with in-degree i and out-degree j in G_n, i.e.,

$$N_n(i, j) = \sum_{v \in V_n} \mathbf{1}_{\{D_n(v)=(i,j)\}} \quad (n \geq 1),$$

and set $\nu_n(i, j) = \mathbf{E}(N_n(i, j))$. An analogue of Lemma 3.1 below has been proved in Ref.[3], where only the marginal in-degree distribution is considered. See also Ref.[16]. We extend the result to the joint distribution of both in- and out-degrees, which extends Theorem 3.2 in Ref.[3] and implies that for each i and j there are non-random constants (p_{ij}) such that

$$\frac{N_n(i, j)}{n} \to p_{ij} \text{ a.s. as } n \to \infty. \tag{3.1}$$

Clearly, $p_{00} = 0$. We also take $N_n(i, j)$ and p_{ij} to be zero if either i or j is -1. The explicit form of the limiting degree distribution (p_{ij}) is given in Ref.[3]. By the model assumptions in Section 2, up to stage n, $i + j \leq n + 1$ for all (i, j), since at each stage we can only increase either the in- or out-degree of one particular node by 1. Therefore, with probability 1, $N_n(i, j) = 0$ for $i + j > n + 1$.

Lemma 3.1. For each $i, j = 0, 1, 2, \ldots,$

$$\max_{(i,j):i+j \leq n+1} |\nu_n(i, j) - np_{ij}| \leq 1, \text{ for } \forall n \geq 1, \tag{3.2}$$

and for any $C > 6$,

$$\mathbf{P}\left(\max_{(i,j):i+j \leq n+1} \left|\frac{N_n(i, j)}{n} - p_{ij}\right| \geq C\sqrt{\frac{\log n}{n}} + \frac{1}{n}\right) = o(1), \text{ as } n \to \infty, \tag{3.3}$$

where the p_{ij} satisfy [3, p. 138, eqn. 6.13]

$$\begin{aligned} p_{ij} = {} & \alpha \mathbf{1}_{\{(i,j)=(0,1)\}} + \gamma \mathbf{1}_{\{(i,j)=(1,0)\}} + c_1(i - 1 + \lambda)p_{i-1,j} \\ & + c_2(j - 1 + \mu)p_{i,j-1} - \delta_{ij}p_{ij}. \end{aligned} \tag{3.4}$$

Here, we have

$$c_1 = \frac{\alpha}{1 + \lambda}, \quad c_2 = \frac{\gamma}{1 + \mu}, \quad \delta_{ij} = c_1(i + \lambda) + c_2(j + \mu). \tag{3.5}$$

As a stochastic process in (i, j), the centered proportion of nodes with in-degree i and out-degree j converges in distribution to a centered Gaussian process. Asymptotic normality relies on a standard multivariate martingale central limit theorem (cf. Proposition 2.2 outlined in Ref.[14]; a statement is given in Proposition 4.1 in Section 4.2 and see also Refs.[4,7,8,10] and Ref.[6, Chap. 8]). For our problem, the normality results are summarized in the next theorem.

Theorem 3.1. *Fix positive integers I, O. In the normality statement, matrices Σ_{IO} and K_{IO} are specified in (4.43) and (4.44), respectively. Provided that K_{IO} is invertible, we have in $\mathbb{R}^{(I+1)(O+1)}$*

$$\left(\sqrt{n}\left(\frac{N_n(i,\,j)}{n} - p_{ij}\right) : \, 0 \le i \le I, \; 0 \le j \le O\right) \Rightarrow N\left(0, K_{IO}^{-1}\Sigma_{IO}K_{IO}^{-T}\right). \tag{3.6}$$

In particular, for non-random constants $\xi_{k,l}^{(i,j)}$ defined as

$$\xi_{k,l}^{(i,j)} := \begin{cases} 1 & \text{if } k = i, \; l = j; \\ (-1)^{i-k}\prod_{d=k}^{i-1}\frac{\lambda+d}{i-d} & \text{if } 0 \le k \le i-1, \; l = j; \\ (-1)^{j-l}\prod_{d=k}^{j-1}\frac{\mu+d}{j-d} & \text{if } k = i, \; 0 \le l \le j-1; \\ -\left(\dfrac{c_1(k+\lambda)}{\delta_{ij}-\delta_{kl}}\xi_{k+1,l}^{(i,j)} + \dfrac{c_2(l+\mu)}{\delta_{ij}-\delta_{kl}}\xi_{k,l+1}^{(i,j)}\right) & \text{if } 0 \le k \le i-1, \\ \quad 0 \le l \le j-1, \; (k,l) \ne (0,0); & \\ 0 & \text{otherwise}, \end{cases}$$

we have

$$Var\left(\sqrt{n}\sum_{k=0}^{i}\sum_{l=0}^{j}\xi_{k,l}^{(i,j)}\left(\frac{N_n(i,\,j)}{n} - p_{ij}\right)\right) \longrightarrow \frac{\varsigma(i,\,j)}{1+2\delta_{ij}}, \quad \text{as } n \to \infty,$$

where

$$\begin{aligned}\varsigma(i,\,j) = {} & (\alpha + c_1\lambda p_{01})\left(\xi_{0,1}^{(i,j)}\right)^2 + (\gamma + c_2\mu p_{10})\left(\xi_{1,0}^{(i,j)}\right)^2 - \left(\sum_{k=0}^{i}\sum_{l=0}^{j}p_{kl}\xi_{k,l}^{(i,j)}\right)^2 \\ & + \sum_{q=0}^{\infty}\sum_{r=0}^{\infty}\left\{c_1(r+\lambda)p_{rq}\left(\xi_{r+1,q}^{(i,j)} + \xi_{0,1}^{(i,j)} - \xi_{r,q}^{(i,j)}\right)^2\right. \\ & \left. + c_2(q+\mu)p_{rq}\left(\xi_{r,q+1}^{(i,j)} + \xi_{1,0}^{(i,j)} - \xi_{r,q}^{(i,j)}\right)^2\right\} \\ & + \left[\sum_{k=0}^{i}\sum_{l=0}^{j}p_{kl}\left((\delta_{kl}+1)\xi_{k,l}^{(i,j)} - c_1(k+\lambda)\xi_{k+1,l}^{(i,j)} - c_2(l+\mu)\xi_{k,l+1}^{(i,j)}\right)\right. \\ & \left. - (\alpha\xi_{0,1}^{(i,j)} + \gamma\xi_{1,0}^{(i,j)})\right]^2.\end{aligned}$$

4. Proofs

4.1. Proof of Lemma 3.1.

By the construction of our model, at the initial stage we have $N_1(1,1) = 1$, $N_1(i,j) = 0$ for $(i,j) \neq (1,1)$. Let $\mathcal{F}_n$ be the σ-field of information accumulated by watching the graph grow until stage n. We have

$$\begin{aligned}
\mathbf{E}(N_{n+1}(i,j)|\mathcal{F}_n) &= N_n(i,j) + \mathbf{E}(N_{n+1}(i,j) - N_n(i,j)|\mathcal{F}_n) \\
&= N_n(i,j) + \alpha \mathbf{1}_{\{(i,j)=(0,1)\}} + \gamma \mathbf{1}_{\{(i,j)=(1,0)\}} \\
&\quad + \mathbf{P}(\text{a new edge from } n+1 \text{ to } v \in V_n; D_n(v) = (i-1,j)|\mathcal{F}_n) \\
&\quad + \mathbf{P}(\text{a new edge from } v \in V_n \text{ to } n+1; D_n(v) = (i,j-1)|\mathcal{F}_n) \\
&\quad - \mathbf{P}(\text{a new edge from } n+1 \text{ to } v \in V_n; D_n(v) = (i,j)|\mathcal{F}_n) \\
&\quad - \mathbf{P}(\text{a new edge from } v \in V_n \text{ to } n+1; D_n(v) = (i,j)|\mathcal{F}_n) \\
&= N_n(i,j) + \alpha \mathbf{1}_{\{(i,j)=(0,1)\}} + \gamma \mathbf{1}_{\{(i,j)=(1,0)\}} \\
&\quad + c_1(i-1+\lambda)\frac{N_n(i-1,j)}{n} + c_2(j-1+\mu)\frac{N_n(i,j-1)}{n} \\
&\quad - (c_1(i+\lambda)) + c_2(j+\mu))\frac{N_n(i,j)}{n}. \qquad (4.1)
\end{aligned}$$

Taking expectations and recalling that $\nu_n(i,j) := \mathbf{E}(N_n(i,j))$, we get

$$\begin{aligned}
\nu_{n+1}(i,j) &= \alpha \mathbf{1}_{\{(i,j)=(0,1)\}} + \gamma \mathbf{1}_{\{(i,j)=(1,0)\}} \\
&\quad + \left(1 - \frac{\delta_{ij}}{n}\right)\nu_n(i,j) + \frac{c_1(i-1+\lambda)}{n}\nu_n(i-1,j) \\
&\quad + \frac{c_2(j-1+\mu)}{n}\nu_n(i,j-1). \qquad (4.2)
\end{aligned}$$

Define $\varepsilon_n(i,j) = \nu_n(i,j) - np_{ij}$. Since $N_n(i,j)/n \to p_{ij}$ a.s. as $n \to \infty$ and $N_n(i,j)/n \leq 1$ for all (i,j), it follows that

$$|\varepsilon_1(1,1)| = |1 - p_{11}| \leq 1, \quad |\varepsilon_1(i,j)| = |0 - p_{ij}| \leq 1 \text{ for } (i,j) \neq (1,1). \quad (4.3)$$

Also, for $n \geq 1$

$$\varepsilon_{n+1}(0,1) = \left(1 - \frac{\delta_{01}}{n}\right)\varepsilon_n(0,1), \quad \varepsilon_{n+1}(1,0) = \left(1 - \frac{\delta_{10}}{n}\right)\varepsilon_n(1,0),$$

and further for $(i,j) \notin \{(0,1),(1,0)\}$

$$\begin{aligned}
\varepsilon_{n+1}(i,j) &= \nu_{n+1}(i,j) - (n+1)p_{ij} \\
&= \left(1 - \frac{\delta_{ij}}{n}\right)\varepsilon_n(i,j) + \frac{c_1(i-1+\lambda)}{n}\varepsilon_n(i-1,j)
\end{aligned}$$

$$+\frac{c_2(j-1+\mu)}{n}\varepsilon_n(i,j-1).$$

Then, (3.2) is true for $(i,j)=(0,1)$ by simple induction on n: if $|\varepsilon_n(0,1)|\le 1$, then

$$|\varepsilon_{n+1}(0,1)|\le\left|1-\frac{\delta_{01}}{n}\right|\le 1,$$

because $\delta_{01}=\frac{\gamma}{1+\mu}(1+\mu)+\frac{\alpha}{1+\lambda}\lambda<\alpha+\gamma=1$. Similar arguments give that (3.2) also holds for $(i,j)=(1,0)$. For $(i,j)\notin\{(0,1),(1,0)\}$, our induction assumption $\max_{(i,j):i+j\le n+1}|\varepsilon_n(i,j)|\le 1$ (which is true for $n=1$ by (4.3)) gives that

$$\begin{aligned}|\varepsilon_{n+1}(i,j)|&=\left|\left(1-\frac{\delta_{ij}}{n}\right)\varepsilon_n(i,j)+\frac{c_1(i-1+\lambda)}{n}\varepsilon_n(i-1,j)\right.\\&\qquad\left.+\frac{c_2(j-1+\mu)}{n}\varepsilon_n(i,j-1)\right|\\&\le\left|1-\frac{\delta_{ij}}{n}+\frac{c_1(i-1+\lambda)}{n}+\frac{c_2(j-1+\mu)}{n}\right|\\&=\left|1-\frac{c_1+c_2}{n}\right|\le 1,\end{aligned}$$

by noting that $c_1+c_2\le\alpha+\gamma=1$. Hence,

$$\begin{aligned}&\max_{(i,j):i+j\le n+1}|\varepsilon_{n+1}(i,j)|\\&\quad=\max\left\{|\varepsilon_{n+1}(0,1)|,|\varepsilon_{n+1}(1,0)|,\max_{(i,j)\notin\{(0,1),(1,0)\}:i+j\le n+1}|\varepsilon_{n+1}(i,j)|\right\}\le 1.\end{aligned}$$

This verifies (3.2).

Next fix (i,j) and n and define the uniformly integrable martingale

$$Y_m(i,j)=\mathbf{E}(N_n(i,j)|\mathcal{F}_m),\qquad m=0,1,\dots,n,$$

with difference sequence

$$d_m(i,j)=Y_m(i,j)-Y_{m-1}(i,j).$$

Given $\mathcal{F}_m$, determining G_n requires the identification of which old vertices are involved at each stage and there are at most $2n$ such choices. Under proper redistribution, changing one of these choices (say from node u to node v) at some stage m will alter the degrees of u and v in the final graph. Also, the future degree of the new node created at stage m will be changed if we switch between "edge from the new node" to "edge to the new node." Hence,

$$|d_m(i,j)|\le 3,\qquad m=0,1,\dots,n.$$

Also, $Y_0(i, j) = \mathbf{E}(N_n(i, j)|\mathcal{F}_0) = \nu_n(i, j)$, and

$$N_n(i, j) - \nu_n(i, j) = \sum_{k=1}^{n} d_k(i, j).$$

Then, by the Azuma–Hoeffding's inequality[1], for any $C > 0$,

$$\mathbf{P}(|N_n(i, j) - \nu_n(i, j)| \geq C\sqrt{n \log n}) \leq 2 \exp\left(-\frac{C^2 n \log n}{2n \cdot 3^2}\right) = \frac{2}{n^{C^2/18}}.$$

Therefore,

$$\begin{aligned}
&\mathbf{P}\left(\max_{(i,j):i+j\leq n+1} |N_n(i, j) - \nu_n(i, j)| \geq C\sqrt{n \log n}\right) \\
&\leq n^2 \max_{(i,j):i+j\leq n+1} \mathbf{P}(|N_n(i, j) - \nu_n(i, j)| \geq C\sqrt{n \log n}) \\
&\leq 2n^{-(C^2/18-2)}.
\end{aligned}$$

In other words, for $C^2/18 - 2 > 0$ or $C > 6$, we have

$$\mathbf{P}\left(\max_{(i,j):i+j\leq n+1} |N_n(i, j) - \nu_n(i, j)| \geq C\sqrt{n \log n}\right) = o(1). \tag{4.4}$$

Now, (3.2) and (4.4) together imply that

$$\mathbf{P}\left(\max_{(i,j):i+j\leq n+1} \left|\frac{N_n(i, j)}{n} - p_{ij}\right| \geq \frac{1}{n}\left(C\sqrt{n \log n} + 1\right)\right) = o(1),$$

and this gives (3.3) for $C > 6$.

4.2. Proof of Theorem 3.1.

4.2.1. Sketch of the proof

The key to prove Theorem 3.1 is to use the martingale central limit theorem, and we include a multivariate version of it in Proposition 4.1. The martingale we are going to consider is of the form

$$M_n(i, j) = \sum_{l=0}^{j} \sum_{k=0}^{i} b_{k,l,n}^{(i,j)} (N_n(k, l) - \nu_n(k, l)),$$

where $b_{k,l,n}^{(i,j)}$ are some non-random constants.

In order to specify the limiting scaling matrix K_{IO}, we first investigate the limits of the ratio $b_{k,l,n}^{(i,j)}/b_{i,j,n}^{(i,j)}$ as $n \to \infty$, and K_{IO} consists of these limits. Next, we compute the asymptotic conditional covariances of properly scaled martingale differences which will lead to the limiting covariance matrix Σ_{IO}, according to Proposition 4.1. Also, we check the two assumptions of Proposition 4.1 in the end to verify that the martingale central limit theorem is applicable here. Then, the concentration inequality proved in Lemma 3.1 allows replacement of expectations by the p_{ij}'s and gives us the desired results.

4.2.2. Preliminary: Martingale central limit theorem

We use a multivariate martingale central limit theorem, Proposition 4.1, to prove Theorem 3.1. See Proposition 2.2 in Ref.[14] and also Refs.[4,7,8,10] and Ref.[6, Chap. 8]).

Proposition 4.1. *Let* $\{\mathbf{X}_{n,m}, \mathcal{G}_{n,m}, 1 \le m \le n\}$, $\mathbf{X}_{n,m} = (X_{n,m,1}, \dots, X_{n,m,d})^T$, *be a d-dimensional square-integrable martingale difference array. Consider the* $d \times d$ *non-negative definite random matrices*

$$G_{n,m} = \left(\mathbf{E}(X_{n,m,i}X_{n,m,j}|\mathcal{G}_{n,m-1}), i, j = 1, 2, \dots, d\right), \quad V_n = \sum_{m=1}^{n} G_{n,m},$$

and suppose (A_n) *is a sequence of* $l \times d$ *matrices with a bounded supremum norm. Assume that*

(i) $A_n V_n A_n^T \xrightarrow{P} \Sigma$ *as* $n \to \infty$ *for some deterministic (automatically non-negatively definite) matrix* Σ.

(ii) $\sum_{m \le n} \mathbf{E}(X_{n,m,i}^2 1_{\{|X_{n,m,i}|>\epsilon\}}|\mathcal{G}_{n,m-1}) \xrightarrow{P} 0$ *as* $n \to \infty$ *for all* $i = 1, 2, \dots, d$ *and* $\epsilon > 0$.

Then, in $\mathbb{R}^l$, *as* $n \to \infty$

$$\sum_{m=1}^{n} A_n \mathbf{X}_{n,m} \Rightarrow \mathbf{X}, \tag{4.5}$$

a centered l-dimensional Gaussian vector with covariance matrix Σ.

4.2.3. The martingale

We start with constructing a martingale for fixed i and j. Suppose that our martingale takes the form

$$M_n(i, j) = \sum_{l=0}^{j} \sum_{k=0}^{i} b_{k,l,n}^{(i,j)} (N_n(k, l) - \nu_n(k, l)), \tag{4.6}$$

where $b_{k,l,n}^{(i,j)}$ are some non-random constants. We investigate what properties $b_{k,l,n}^{(i,j)}$ must satisfy in order that $M_n(i, j)$ is a martingale in the index n.

Using (4.1) and (4.2), we see that in order to make $M_n(i, j)$ a martingale, we must have

$$\begin{aligned}\mathbf{E}(M_{n+1}(i, j)|\mathcal{F}_n) = \sum_{l=0}^{j} \sum_{k=0}^{i} b_{k,l,n+1}^{(i,j)} \Bigg[&\left(1 - \frac{\delta_{kl}}{n}\right) (N_n(k, l) - \nu_n(k, l)) \\ &+ \frac{c_1(k - 1 + \lambda)}{n} (N_n(k - 1, l) - \nu_n(k - 1, l)) \\ &+ \frac{c_2(l - 1 + \mu)}{n} (N_n(k, l - 1) - \nu_n(k, l - 1)) \Bigg]\end{aligned}$$

$$= \sum_{l=0}^{j} \sum_{k=0}^{i} b_{k,l,n}^{(i,j)} (N_n(k,l) - \nu_n(k,l)) = M_n(i,j),$$

where the last equality follows from the martingale assumption. Thus, it is sufficient for the coefficients $b_{k,l,n}^{(i,j)}$, $0 \le k \le i$, $0 \le l \le j$ to satisfy the following recursions:

$$b_{i,j,n+1}^{(i,j)} \left(1 - \frac{\delta_{ij}}{n}\right) = b_{i,j,n}^{(i,j)}, \tag{4.7}$$

$$b_{k,j,n+1}^{(i,j)} \left(1 - \frac{\delta_{kj}}{n}\right) + b_{k+1,j,n+1}^{(i,j)} \frac{c_1(k+\lambda)}{n} = b_{k,j,n}^{(i,j)}, \quad 0 \le k \le i-1, \tag{4.8}$$

$$b_{i,l,n+1}^{(i,j)} \left(1 - \frac{\delta_{il}}{n}\right) + b_{i,l+1,n+1}^{(i,j)} \frac{c_2(l+\mu)}{n} = b_{i,l,n}^{(i,j)}, \quad 0 \le l \le j-1, \tag{4.9}$$

$$b_{k,l,n+1}^{(i,j)} \left(1 - \frac{\delta_{kl}}{n}\right) + b_{k+1,l,n+1}^{(i,j)} \frac{c_1(k+\lambda)}{n} + b_{k,l+1,n+1}^{(i,j)} \frac{c_2(l+\mu)}{n} = b_{k,l,n}^{(i,j)},$$
$$0 \le k \le i-1, 0 \le l \le j-1. \tag{4.10}$$

The recursions (4.7)–(4.10) will not have a straightforward solution if $\delta_{ij} = m$ for some $m < n$. If $\delta_{ij} = m$, then from the definition of δ_{ij} in (3.5), we have

$$\begin{aligned} m = \delta_{ij} &= \frac{\alpha}{1+\lambda} i + \alpha - \frac{\alpha}{1+\lambda} + \frac{\gamma}{1+\mu} j + \gamma - \frac{\gamma}{1+\mu} \\ &= \frac{\alpha}{1+\lambda}(i-1) + \frac{\gamma}{1+\mu}(j-1) + 1 < i + j - 1. \end{aligned}$$

However, a particular node v with in-degree i and out-degree j cannot exist before stage $i + j - 1$ and therefore, almost surely, $N_m(i,j) = 0$ for all $m < i + j - 1$. The same arguments hold for all $0 \le k \le i$, $0 \le l \le j$ relevant for (4.8) and (4.9). Hence, for solving the recursions, we set $b_{k,l,m}^{(i,j)} = 0$ for all $m < k + l - 1$ and $b_{k,l,k+l-1}^{(i,j)} = 1$.

Solving (4.7) gives

$$b_{i,j,n+1}^{(i,j)} = \prod_{m=i+j-1}^{n} \left(1 - \frac{\delta_{ij}}{m}\right)^{-1}. \tag{4.11}$$

Also, (4.8) yields

$$\begin{aligned} b_{k,j,n+1}^{(i,j)} &= \left(1 - \frac{\delta_{kj}}{n}\right)^{-1} b_{k,j,n}^{(i,j)} - \left(1 - \frac{\delta_{kj}}{n}\right)^{-1} b_{k+1,j,n+1}^{(i,j)} \frac{c_1(k+\lambda)}{n} \\ &= \left(1 - \frac{\delta_{kj}}{n}\right)^{-1} \left[\left(1 - \frac{\delta_{kj}}{n-1}\right)^{-1} b_{k,j,n-1}^{(i,j)} \right. \\ &\quad \left. - \left(1 - \frac{\delta_{kj}}{n-1}\right)^{-1} b_{k+1,j,n}^{(i,j)} \frac{c_1(k+\lambda)}{n-1} \right] \\ &\quad - \left(1 - \frac{\delta_{kj}}{n}\right)^{-1} b_{k+1,j,n+1}^{(i,j)} \frac{c_1(k+\lambda)}{n} \end{aligned}$$

$$= \cdots = \prod_{m=k+j-1}^{n} \left(1 - \frac{\delta_{kj}}{m}\right)^{-1}$$
$$- \sum_{m=k+j-1}^{n} b_{k+1,j,m+1}^{(i,j)} \frac{c_1(k+\lambda)}{m} \prod_{d=m}^{n} \left(1 - \frac{\delta_{kj}}{d}\right)^{-1}. \tag{4.12}$$

Similarly, we obtain from (4.9) and (4.10) that

$$b_{i,l,n+1}^{(i,j)} = \prod_{m=i+l-1}^{n} \left(1 - \frac{\delta_{il}}{m}\right)^{-1} - \sum_{m=i+l-1}^{n} b_{i,l+1,m+1}^{(i,j)} \frac{c_2(l+\mu)}{m} \prod_{d=m}^{n} \left(1 - \frac{\delta_{il}}{d}\right)^{-1},$$

and that

$$b_{k,l,n+1}^{(i,j)} = \prod_{m=k+l-1}^{n} \left(1 - \frac{\delta_{kl}}{m}\right)^{-1} - \sum_{m=k+l-1}^{n} b_{k+1,l,m+1}^{(i,j)} \frac{c_1(k+\lambda)}{m} \prod_{d=m}^{n} \left(1 - \frac{\delta_{kl}}{d}\right)^{-1}$$
$$- \sum_{m=k+l-1}^{n} b_{k,l+1,m+1}^{(i,j)} \frac{c_2(l+\mu)}{m} \prod_{d=m}^{n} \left(1 - \frac{\delta_{kl}}{d}\right)^{-1}. \tag{4.13}$$

4.2.4. Properties of the coefficients $b_{k,l,n+1}^{(i,j)}$

For the calculation of the asymptotic form of conditional covariances of martingale differences, we will need the asymptotic forms of the ratio $b_{k,l,n+1}^{(i,j)}/b_{i,j,n+1}^{(i,j)}$ for all $k \le i$, $l \le j$, as $n \to \infty$ and we set

$$\xi_{k,l}^{(i,j)} := \lim_{n\to\infty} \frac{b_{k,l,n+1}^{(i,j)}}{b_{i,j,n+1}^{(i,j)}}, \quad k = 0, 1, \ldots, i \text{ and } l = 0, 1, \ldots, j. \tag{4.14}$$

We begin with the case $l = j$. Using (4.11) and (4.12), we know that for $0 \le k \le i-1$,

$$\frac{b_{k,j,n+1}^{(i,j)}}{b_{i,j,n+1}^{(i,j)}} = \frac{\prod_{m=k+j-1}^{n} \left(1 - \frac{\delta_{kj}}{m}\right)^{-1}}{\prod_{m=i+j-1}^{n} \left(1 - \frac{\delta_{ij}}{m}\right)^{-1}}$$
$$- \sum_{m=k+j-1}^{n} \frac{b_{k+1,j,m+1}^{(i,j)}}{m} c_1(k+\lambda) \frac{\prod_{d=m}^{n} \left(1 - \frac{\delta_{kj}}{d}\right)^{-1}}{\prod_{d=i+j-1}^{n} \left(1 - \frac{\delta_{ij}}{d}\right)^{-1}}, \tag{4.15}$$

and from (4.15), we claim that

$$\frac{b^{(i,j)}_{k,j,n+1}}{b^{(i,j)}_{i,j,n+1}} \to \xi^{(i,j)}_{k,j} = (-1)^{i-k} \prod_{d=k}^{i-1} \left(\frac{\lambda + d}{i - d} \right). \tag{4.16}$$

For the first term on the right of (4.15), we have by Stirling's formula

$$\begin{aligned}
\frac{\prod_{m=k+j-1}^{n} \left(1 - \frac{\delta_{kj}}{m}\right)^{-1}}{\prod_{m=i+j-1}^{n} \left(1 - \frac{\delta_{ij}}{m}\right)^{-1}} &= \prod_{m=k+j-1}^{i+j-2} \frac{m}{m - \delta_{kj}} \times \prod_{m=i+j-1}^{n} \frac{m - \delta_{ij}}{m - \delta_{kj}} \\
&= \frac{\Gamma(i+j-1)/\Gamma(k+j-1)}{\Gamma(i+j-1-\delta_{kj})/\Gamma(k+j-1-\delta_{kj})} \\
&\quad \times \frac{\Gamma(n+1-\delta_{ij})/\Gamma(i+j-1-\delta_{ij})}{\Gamma(n+1-\delta_{kj})/\Gamma(i+j-1-\delta_{kj})} \\
&= \frac{\Gamma(n+1-\delta_{ij})}{\Gamma(n+1-\delta_{kj})} \frac{\Gamma(k+j-\delta_{kj})}{\Gamma(i+j-\delta_{ij})} \frac{\Gamma(i+j-1)}{\Gamma(k+j-1)} \\
&\sim n^{-(i-k)c_1} \frac{\Gamma(k+j-\delta_{kj})}{\Gamma(i+j-\delta_{ij})} \frac{\Gamma(i+j-1)}{\Gamma(k+j-1)} \to 0, \qquad (4.17)
\end{aligned}$$

as $n \to \infty$, because $i - k \geq 1$. Hence, proving (4.16) requires showing

$$-\sum_{m=k+j-1}^{n} \frac{b^{(i,j)}_{k+1,j,m+1}}{b^{(i,j)}_{i,j,n+1}} \frac{c_1(k+\lambda)}{m} \prod_{d=m}^{n} \left(1 - \frac{\delta_{kj}}{d}\right)^{-1} \to (-1)^{i-k} \prod_{d=k}^{i-1} \left(\frac{\lambda + d}{i - d}\right), \tag{4.18}$$

and we prove this by induction on $k < i$. For $k = i - 1$, using (4.11), we have

$$\begin{aligned}
&-\sum_{m=i+j-2}^{n} \frac{b^{(i,j)}_{i,j,m+1}}{b^{(i,j)}_{i,j,n+1}} \frac{c_1(i-1+\lambda)}{m} \prod_{d=m}^{n} \left(1 - \frac{\delta_{i-1,j}}{d}\right)^{-1} \\
&= -c_1(i-1+\lambda) \sum_{m=i+j-2}^{n} \frac{1}{m} \frac{\prod_{d=m}^{n} \left(1 - \frac{\delta_{i-1,j}}{d}\right)^{-1}}{\prod_{d=m+1}^{n} \left(1 - \frac{\delta_{ij}}{d}\right)^{-1}} \\
&= -c_1(i-1+\lambda) \sum_{m=i+j-2}^{n} \frac{\Gamma(n+1-\delta_{ij})/\Gamma(m+1-\delta_{ij})}{\Gamma(n+1-\delta_{i-1,j})/\Gamma(m-\delta_{i-1,j})} \\
&= -c_1(i-1+\lambda) \frac{\Gamma(n+1-\delta_{ij})}{\Gamma(n+1-\delta_{i-1,j})} \sum_{m=i+j-2}^{n} g(m), \qquad (4.19)
\end{aligned}$$

where

$$g(m) = \frac{\Gamma(m - \delta_{i-1,j})}{\Gamma(m + 1 - \delta_{ij})}.$$

Stirling's formula gives as $n \to \infty$,

$$\frac{\Gamma(n + 1 - \delta_{ij})}{\Gamma(n + 1 - \delta_{i-1,j})} \sim n^{\delta_{i-1,j} - \delta_{ij}} = n^{-c_1},$$

and also

$$g(n) \sim n^{c_1 - 1}, \quad (n \to \infty).$$

So the function $g(n)$ is regularly varying and hence by Karamata's theorem on integration (see, for example, Ref.[12]), we have

$$\sum_{m=i+j-2}^{n} g(m) \sim n^{c_1}/c_1,$$

and thus (4.19) is asymptotic to

$$-c_1(i - 1 + \lambda)n^{-c_1} n^{c_1}/c_1 = -(i - 1 + \lambda).$$

This verifies the base case for (4.18) and thus (4.16) is also true when $k = i - 1$.

For the next step in the induction argument, we suppose that (4.18) holds for k. Then, because of (4.17), the claim in (4.16) holds for $k - 1$. We then evaluate the left side of (4.18) with $k + 1$ replaced by $k - 1$. Using (4.11), $\Gamma(t + 1) = t\Gamma(t)$ and calculations similar to what was just done, we get

$$\begin{aligned}
&- \sum_{m=k+j-3}^{n} \frac{b^{(i,j)}_{k-1,j,m+1}}{b^{(i,j)}_{i,j,n+1}} \frac{c_1(k - 2 + \lambda)}{m} \prod_{d=m}^{n} \left(1 - \frac{\delta_{k-2,j}}{d}\right)^{-1} \\
&\quad = - \sum_{m=i+j-1}^{n} \frac{b^{(i,j)}_{k-1,j,m+1}}{b^{(i,j)}_{i,j,m+1}} \frac{c_1(k - 2 + \lambda)}{m} \frac{\prod_{d=m}^{n} \left(1 - \frac{\delta_{k-2,j}}{d}\right)^{-1}}{\prod_{d=m+1}^{n} \left(1 - \frac{\delta_{ij}}{d}\right)^{-1}} \\
&\qquad - \sum_{m=k+j-3}^{i+j-2} b^{(i,j)}_{k-1,j,m+1} \frac{c_1(k - 2 + \lambda)}{m} \prod_{d=m}^{i+j-2} \left(1 - \frac{\delta_{k-2,j}}{d}\right)^{-1} \\
&\qquad \times \prod_{d=i+j-1}^{n} \frac{\left(1 - \frac{\delta_{k-2,j}}{d}\right)^{-1}}{\left(1 - \frac{\delta_{ij}}{d}\right)^{-1}} \\
&\quad = -c_1(k - 2 + \lambda) \frac{\Gamma(n + 1 - \delta_{ij})}{\Gamma(n + 1 - \delta_{k-2,j})} \left(\sum_{m=i+j-1}^{n} h(m) \right.
\end{aligned}$$

$$+\frac{\Gamma(i+j-1)}{\Gamma(i+j-1-\delta_{ij})}\sum_{m=k+j-3}^{i+j-2} b_{k-1,j,m+1}^{(i,j)}\frac{\Gamma(m-\delta_{k-2,j})}{\Gamma(m+1)}\Bigg), \quad (4.20)$$

where

$$h(m)=\frac{b_{k-1,j,m+1}^{(i,j)}}{b_{i,j,m+1}^{(i,j)}}\frac{c_1(k-2+\lambda)}{m}\frac{\prod_{d=m}^{n}\left(1-\frac{\delta_{k-2,j}}{d}\right)^{-1}}{\prod_{d=m+1}^{n}\left(1-\frac{\delta_{ij}}{d}\right)^{-1}}\frac{\Gamma(n+1-\delta_{k-2,j})}{\Gamma(n+1-\delta_{ij})}$$

$$=\frac{b_{k-1,j,m+1}^{(i,j)}}{b_{i,j,m+1}^{(i,j)}}\times\frac{\Gamma(m+1-\delta_{k-2,j})}{\Gamma(m+1-\delta_{ij})}\frac{1}{m-\delta_{k-2,j}}. \quad (4.21)$$

Since the induction assumption means that (4.16) holds for k, we have, as $m\to\infty$, that h is regularly varying with index $(i-k+2)c_1-1$, i.e.,

$$h(m)\sim m^{(i-k+2)c_1-1}(-1)^{i-k+1}\prod_{d=k-1}^{i-1}\left(\frac{\lambda+d}{i-d}\right). \quad (4.22)$$

Again, using Karamata's theorem, we have from (4.20)

$$-c_1(k-2+\lambda)\frac{\Gamma(n+1-\delta_{ij})}{\Gamma(n+1-\delta_{k-2,j})}\sum_{m=i+j-1}^{n}h(m)$$

$$\longrightarrow(-1)^{i-k+2}\frac{k-2+\lambda}{i-k+2}\prod_{d=k-1}^{i-1}\left(\frac{\lambda+d}{i-d}\right)=(-1)^{i-k+2}\prod_{d=k-2}^{i-1}\left(\frac{\lambda+d}{i-d}\right).$$

Also, note that the second term in the bracket in (4.20) is finite and $\Gamma(n+1-\delta_{ij})/\Gamma(n+1-\delta_{k-2,j})\sim n^{-(i-k+2)c_1}\to 0$ as $n\to\infty$ by Stirling's formula. Hence, we conclude that (4.18) holds for all $k=0,1,\ldots,i-1$. With $\xi_{k,j}^{(i,j)}$ defined in (4.14), we have verified (4.16).

Similarly, as $n\to\infty$,

$$\frac{b_{i,l,n+1}^{(i,j)}}{b_{i,j,n+1}^{(i,j)}}\to(-1)^{j-l}\prod_{r=k}^{j-1}\left(\frac{\mu+r}{j-r}\right)=:\xi_{i,l}^{(i,j)}. \quad (4.23)$$

For $0\le k\le i-1$, $0\le l\le j-1$ and $(k,l)\neq(0,0)$, we claim that

$$\frac{b_{k,l,n+1}^{(i,j)}}{b_{i,j,n+1}^{(i,j)}}\sim-\left[\frac{c_1(k+\lambda)}{\delta_{ij}-\delta_{kl}}\times\frac{b_{k+1,l,n+1}^{(i,j)}}{b_{i,j,n+1}^{(i,j)}}+\frac{c_2(l+\mu)}{\delta_{ij}-\delta_{kl}}\times\frac{b_{k,l+1,n+1}^{(i,j)}}{b_{i,j,n+1}^{(i,j)}}\right]$$

$$\to-\left[\frac{c_1(k+\lambda)}{\delta_{ij}-\delta_{kl}}\times\xi_{k+1,l}^{(i,j)}+\frac{c_2(l+\mu)}{\delta_{ij}-\delta_{kl}}\times\xi_{k,l+1}^{(i,j)}\right]=:\xi_{k,l}^{(i,j)}. \quad (4.24)$$

Recall (4.13), and following a similar argument as in (4.17), we conclude that

$$\frac{\prod_{m=k+l-1}^{n}\left(1-\frac{\delta_{kl}}{m}\right)^{-1}}{\prod_{m=i+j-1}^{n}\left(1-\frac{\delta_{ij}}{m}\right)^{-1}} \longrightarrow 0, \quad \text{as } n\to\infty.$$

Also, the same arguments as in (4.20), (4.21) and (4.22) give

$$\begin{aligned}
&- \sum_{m=k+l-1}^{n} \frac{b_{k+1,l,m+1}^{(i,j)}}{b_{i,j,n+1}^{(i,j)}} \frac{c_1(k+\lambda)}{m} \prod_{d=m}^{n}\left(1-\frac{\delta_{kl}}{d}\right)^{-1} \\
&= \quad - \sum_{m=i+j-1}^{n} \frac{b_{k+1,l,m+1}^{(i,j)}}{b_{i,j,m+1}^{(i,j)}} \frac{c_1(k+\lambda)}{m} \frac{\prod_{d=m}^{n}\left(1-\frac{\delta_{kl}}{d}\right)^{-1}}{\prod_{d=m+1}^{n}\left(1-\frac{\delta_{ij}}{d}\right)^{-1}} \\
&\quad - \sum_{m=k+l-1}^{i+j-2} b_{k+1,l,m+1}^{(i,j)} \frac{c_1(k+\lambda)}{m} \prod_{d=m}^{i+j-2}\left(1-\frac{\delta_{kl}}{d}\right)^{-1} \prod_{d=i+j-1}^{n} \frac{\left(1-\frac{\delta_{kl}}{d}\right)^{-1}}{\left(1-\frac{\delta_{ij}}{d}\right)^{-1}} \\
&= -c_1(k+\lambda)\frac{\Gamma(n+1-\delta_{ij})}{\Gamma(n+1-\delta_{kl})}\left(\sum_{m=i+j-1}^{n} H(m)\right. \\
&\qquad \left. + \frac{\Gamma(i+j-1)}{\Gamma(i+j-1-\delta_{ij})} \sum_{m=k+l-1}^{i+j-2} b_{k+1,l,m+1}^{(i,j)} \frac{\Gamma(m-\delta_{kl})}{\Gamma(m+1)}\right),
\end{aligned}$$

where

$$\begin{aligned}
H(m) &= \frac{b_{k+1,l,m+1}^{(i,j)}}{b_{i,j,m+1}^{(i,j)}} \times \frac{\Gamma(m+1-\delta_{kl})}{\Gamma(m+1-\delta_{ij})} \frac{1}{m-\delta_{kl}} \\
&\sim m^{\delta_{ij}-\delta_{kl}-1}\xi_{k+1,l}^{(i,j)}, \qquad \text{as } m\to\infty.
\end{aligned}$$

Using Karamata's theorem, we have as $n\to\infty$

$$-c_1(k+\lambda)\frac{\Gamma(n+1-\delta_{ij})}{\Gamma(n+1-\delta_{kl})} \sum_{m=i+j-1}^{n} H(m) \to -\frac{c_1(k+\lambda)}{\delta_{ij}-\delta_{kl}}\xi_{k+1,l}^{(i,j)}.$$

Meanwhile,

$$\begin{aligned}
&-c_1(k+\lambda)\frac{\Gamma(n+1-\delta_{ij})}{\Gamma(n+1-\delta_{kl})}\frac{\Gamma(i+j-1)}{\Gamma(i+j-1-\delta_{ij})} \sum_{m=k+l-1}^{i+j-2} b_{k+1,l,m+1}^{(i,j)} \frac{\Gamma(m-\delta_{kl})}{\Gamma(m+1)} \\
&\sim -c_1(k+\lambda)n^{-(\delta_{ij}-\delta_{kl})}\frac{\Gamma(i+j-1)}{\Gamma(i+j-1-\delta_{ij})} \sum_{m=k+l-1}^{i+j-2} b_{k+1,l,m+1}^{(i,j)} \frac{\Gamma(m-\delta_{kl})}{\Gamma(m+1)} \to 0.
\end{aligned}$$

Therefore, we conclude that

$$-\sum_{m=k+l-1}^{n} \frac{b_{k+1,l,m+1}^{(i,j)}}{b_{i,j,n+1}^{(i,j)}} \frac{c_1(k+\lambda)}{m} \prod_{d=m}^{n} \left(1-\frac{\delta_{kl}}{d}\right)^{-1} \to -\frac{c_1(k+\lambda)}{\delta_{ij}-\delta_{kl}} \xi_{k+1,l}^{(i,j)}.$$

Similarly,

$$-\sum_{m=k+l-1}^{n} \frac{b_{k,l+1,m+1}^{(i,j)}}{b_{i,j,n+1}^{(i,j)}} \frac{c_2(l+\mu)}{m} \prod_{d=m}^{n} \left(1-\frac{\delta_{kl}}{d}\right)^{-1} \to -\frac{c_2(l+\mu)}{\delta_{ij}-\delta_{kl}} \xi_{k,l+1}^{(i,j)}.$$

Hence, the claim in (4.24) is verified.

We set $\xi_{i,j}^{(i,j)} = 1$, $\xi_{0,0}^{(i,j)} = 0$, and note that $\xi_{k,l}^{(i,j)} = 0$ if either $k > i$ or $l > j$.

4.2.5. Martingale differences

Now we are ready to consider the martingale difference:

$$\begin{aligned} &M_{n+1}(i,j) - M_n(i,j) \\ &= \sum_{l=0}^{j}\sum_{k=0}^{i} \left(b_{k,l,n+1}^{(i,j)} N_{n+1}(k,l) - b_{k,l,n}^{(i,j)} N_n(k,l)\right) \\ &\quad - \sum_{l=0}^{j}\sum_{k=0}^{i} \left(b_{k,l,n+1}^{(i,j)} \nu_{n+1}(k,l) - b_{k,l,n}^{(i,j)} \nu_n(k,l)\right). \end{aligned} \tag{4.25}$$

Consider the second double sum on the right side of (4.25). Recall that $\nu_n(i,j)$ satisfies the recursion in (4.2), and this together with the properties of $b_{k,l,n}^{(i,j)}$ in (4.7)–(4.10) give

$$\begin{aligned} &\sum_{l=0}^{j}\sum_{k=0}^{i} \left(b_{k,l,n+1}^{(i,j)} \nu_{n+1}(k,l) - b_{k,l,n}^{(i,j)} \nu_n(k,l)\right) \\ &= \sum_{l=0}^{j}\sum_{k=0}^{i} \Bigg[b_{k,l,n+1}^{(i,j)} \Bigg(\left(1-\frac{\delta_{kl}}{n}\right) \nu_n(k,l) + c_1(k-1+\lambda)\frac{\nu_n(k-1,l)}{n} \\ &\quad + c_2(l-1+\mu)\frac{\nu_n(k,l-1)}{n} + \alpha \mathbf{1}_{\{(i,j)=(0,1)\}} + \gamma \mathbf{1}_{\{(i,j)=(1,0)\}} \Bigg) \\ &\quad - b_{k,l,n}^{(i,j)} \nu_n(k,l) \Bigg] \end{aligned}$$

and identifying summands corresponding to $(k,l) = (i,j)$, $(k,l) = (i-1,j)$, $(k,l) = (i,j-1)$ and then the rest down to $(k,l) = (0,1)$, $(k,l) = (1,0)$ yields

$$\begin{aligned} &= b_{i,j,n+1}^{(i,j)} \Bigg(\nu_n(i,j)\left(1-\frac{\delta_{ij}}{n}\right) + c_1(i-1+\lambda)\frac{\nu_n(i-1,j)}{n} \\ &\quad + c_2(j-1+\mu)\frac{\nu_n(i,j-1)}{n} \Bigg) \end{aligned}$$

$$
\begin{aligned}
&- b^{(i,j)}_{i,j,n}\nu_n(i,j) \\
&+ b^{(i,j)}_{i-1,j,n+1}\left(\nu_n(i-1,j)\left(1-\frac{\delta_{i-1,j}}{n}\right)+c_1(i-2+\lambda)\frac{\nu_n(i-2,j)}{n}\right. \\
&\left.+ c_2(j-1+\mu)\frac{\nu_n(i-1,j-1)}{n}\right) - b^{(i,j)}_{i-1,j,n}\nu_n(i-1,j) \\
&+ b^{(i,j)}_{i,j-1,n+1}\left(\nu_n(i,j-1)\left(1-\frac{\delta_{i,j-1}}{n}\right)+c_1(i-1+\lambda)\frac{\nu_n(i-1,j-1)}{n}\right. \\
&\left.+ c_2(j-2+\mu)\frac{\nu_n(i,j-2)}{n}\right) - b^{(i,j)}_{i,j-1,n}\nu_n(i,j-1) \\
&+ \cdots + b^{(i,j)}_{0,1,n+1}\left(\alpha+\nu_n(0,1)\left(1-\frac{\delta_{01}}{n}\right)\right) - b^{(i,j)}_{0,1,n}\nu_n(0,1) \\
&+ b^{(i,j)}_{1,0,n+1}\left(\gamma+\nu_n(1,0)\left(1-\frac{\delta_{10}}{n}\right)\right) - b^{(i,j)}_{1,0,n}\nu_n(1,0) \\
= \alpha b^{(i,j)}_{0,1,n+1} &+ \gamma b^{(i,j)}_{1,0,n+1} + \Bigg[\underbrace{\nu_n(i,j)\left(b^{(i,j)}_{i,j,n+1}\left(1-\frac{\delta_{ij}}{n}\right)-b^{(i,j)}_{i,j,n}\right)}_{=0 \text{ by } (4.7)} \\
&+ \sum_{k=0}^{i-1}\nu_n(k,j)\underbrace{\left(b^{(i,j)}_{k,j,n+1}\left(1-\frac{\delta_{kj}}{n}\right)+b^{(i,j)}_{k+1,j,n+1}\frac{c_1(k+\lambda)}{n}-b^{(i,j)}_{k,j,n}\right)}_{=0 \text{ by } (4.8)} \\
&+ \sum_{l=0}^{j-1}\nu_n(i,l)\underbrace{\left(b^{(i,j)}_{i,l,n+1}\left(1-\frac{\delta_{il}}{n}\right)+b^{(i,j)}_{i,l+1,n+1}\frac{c_2(l+\mu)}{n}-b^{(i,j)}_{i,l,n}\right)}_{=0 \text{ by } (4.9)} \\
&+ \sum_{l=0}^{j-1}\sum_{k=0}^{i-1} \\
&\times \underbrace{\left(b^{(i,j)}_{k,l,n+1}\left(1-\frac{\delta_{kl}}{n}\right)+b^{(i,j)}_{k+1,l,n+1}\frac{c_1(k+\lambda)}{n}+b^{(i,j)}_{k,l+1,n+1}\frac{c_2(l+\mu)}{n}-b^{(i,j)}_{k,l,n}\right)}_{=0 \text{ by } (4.10)}\Bigg] \\
= \alpha b^{(i,j)}_{0,1,n+1} &+ \gamma b^{(i,j)}_{1,0,n+1}.
\end{aligned}
$$

So, (4.25) now becomes

$$
\begin{aligned}
M_{n+1}(i,j) - M_n(i,j) = &\sum_{l=0}^{j}\sum_{k=0}^{i}\left(b^{(i,j)}_{k,l,n+1}N_{n+1}(k,l) - b^{(i,j)}_{k,l,n}N_n(k,l)\right) \\
&- \left(\alpha b^{(i,j)}_{0,1,n+1} + \gamma b^{(i,j)}_{1,0,n+1}\right). \qquad (4.26)
\end{aligned}
$$

4.2.6. Conditional covariances

In order to use Proposition 4.1, the multivariate martingale central limit theorem, we need to calculate the asymptotic form of the following quantity:

$$\mathbf{E}\left[\left(\frac{M_{n+1}(i,j)-M_n(i,j)}{\prod_{d=i+j-1}^{n}\left(1-\frac{\delta_{ij}}{d}\right)^{-1}}\right)\left(\frac{M_{n+1}(s,t)-M_n(s,t)}{\prod_{d=s+t-1}^{n}\left(1-\frac{\delta_{st}}{d}\right)^{-1}}\right)\middle|\mathcal{F}_n\right], \tag{4.27}$$

for fixed pairs (i,j) and (s,t). From (4.26), we know that we need to consider in particular

$$\begin{aligned}
&b_{k,l,n+1}^{(i,j)}N_{n+1}(k,l)-b_{k,l,n}^{(i,j)}N_n(k,l)\\
&\quad= b_{k,l,n+1}^{(i,j)}N_{n+1}(k,l)\\
&\qquad-\left(b_{k,l,n+1}^{(i,j)}\left(1-\frac{\delta_{kl}}{n}\right)+b_{k+1,l,n+1}^{(i,j)}\frac{c_1(k+\lambda)}{n}+b_{k,l+1,n+1}^{(i,j)}\frac{c_2(l+\mu)}{n}\right)N_n(k,l)
\end{aligned}$$

(where we applied (4.10))

$$\begin{aligned}
&= b_{k,l,n+1}^{(i,j)}(N_{n+1}(k,l)-N_n(k,l))\\
&\quad+\frac{N_n(k,l)}{n}\left(\delta_{kl}b_{k,l,n+1}^{(i,j)}-c_1(k+\lambda)b_{k+1,l,n+1}^{(i,j)}-c_2(l+\mu)b_{k,l+1,n+1}^{(i,j)}\right).
\end{aligned} \tag{4.28}$$

Recall (3.1) gives $N_n(k,l)/n \to p_{kl}$ a.s. as $n\to\infty$. So dealing with (4.27) means we must calculate the asymptotic form of the conditional moments of

$$\Delta_{n+1}(i,j) := N_{n+1}(i,j)-N_n(i,j).$$

Observe that

$$\Delta_{n+1}(0,1)=\begin{cases}1 & \text{w.p. } \alpha,\\ -1 & \text{w.p. } \delta_{01}\frac{N_n(0,1)}{n},\\ 0 & \text{otherwise;}\end{cases} \tag{4.29}$$

$$\Delta_{n+1}(1,0)=\begin{cases}1 & \text{w.p. } \gamma,\\ -1 & \text{w.p. } \delta_{10}\frac{N_n(1,0)}{n},\\ 0 & \text{otherwise;}\end{cases} \tag{4.30}$$

$$\Delta_{n+1}(k,l)=\begin{cases}1 & \text{w.p. } c_1(k-1+\lambda)\frac{N_n(k-1,l)}{n}+c_2(l-1+\mu)\frac{N_n(k,l-1)}{n},\\ -1 & \text{w.p. } \delta_{kl}\frac{N_n(k,l)}{n},\\ 0 & \text{otherwise,}\end{cases} \tag{4.31}$$

for $(k,l)\notin\{(0,1),(1,0)\}$. For instance, to justify (4.29), we create a $(0,1)$-node when node $n+1$ is born and attaches to V_n but we destroy a $(0,1)$-node if either $n+1$ is born and attaches to a $(0,1)$-node or $v\in V_n$ attaches to $n+1$ and has degree $(0,1)$. Then, using (3.1), (3.4) and (4.29)–(4.31), for each pair (k,l), we have

$$\mathbf{E}(\Delta_{n+1}(k,l)|\mathcal{F}_n)\to p_{kl}, \quad \text{a.s. as } n\to\infty. \tag{4.32}$$

Therefore, from (4.11)

$$\frac{M_{n+1}(i,\,j)-M_n(i,\,j)}{\prod_{d=i+j-1}^{n}\left(1-\frac{\delta_{ij}}{d}\right)^{-1}}=\frac{M_{n+1}(i,\,j)-M_n(i,\,j)}{b_{i,\,j,n+1}^{(i,\,j)}}$$

and applying (4.26) and then (4.28), we have as $n\to\infty$

$$=\sum_{l=0}^{j}\sum_{k=0}^{i}\left[\frac{N_n(k,\,l)}{n}\left(\delta_{kl}\frac{b_{k,l,n+1}^{(i,\,j)}}{b_{i,\,j,n+1}^{(i,\,j)}}-c_1(k+\lambda)\frac{b_{k+1,l,n+1}^{(i,\,j)}}{b_{i,\,j,n+1}^{(i,\,j)}}-c_2(l+\mu)\frac{b_{k,l+1,n+1}^{(i,\,j)}}{b_{i,\,j,n+1}^{(i,\,j)}}\right)\right.$$
$$\left.+\frac{b_{k,l,n+1}^{(i,\,j)}}{b_{i,\,j,n+1}^{(i,\,j)}}\Delta_{n+1}(k,\,l)\right]-\frac{(\alpha b_{0,1,n+1}^{(i,\,j)}+\gamma b_{1,0,n+1}^{(i,\,j)})}{b_{i,\,j,n+1}^{(i,\,j)}}.$$

By the fact that $N_n(k,\,l)/n\to p_{kl}$ a.s. as $n\to\infty$, the above is asymptotically equivalent to

$$\sum_{l=0}^{j}\sum_{k=0}^{i}p_{kl}\left(\delta_{kl}\xi_{k,l}^{(i,\,j)}-c_1(k+\lambda)\xi_{k+1,l}^{(i,\,j)}-c_2(l+\mu)\xi_{k,l+1}^{(i,\,j)}\right)+\xi_{k,l}^{(i,\,j)}\Delta_{n+1}(k,\,l)$$
$$-\left(\alpha\xi_{0,1}^{(i,\,j)}+\gamma\xi_{1,0}^{(i,\,j)}\right),$$

according to (3.1) and definition of $\xi_{k,l}^{(i,\,j)}$ given in (4.14), (4.23) and (4.24).

Recall (4.32), we see that as $n\to\infty$ the conditional expectation in (4.27) is equivalent to

$$\mathbf{E}\left\{\left[\sum_{l=0}^{j}\sum_{k=0}^{i}\left\{p_{kl}\left(\delta_{kl}\xi_{k,l}^{(i,\,j)}-c_1(k+\lambda)\xi_{k+1,l}^{(i,\,j)}-c_2(l+\mu)\xi_{k,l+1}^{(i,\,j)}\right)\right.\right.\right.$$
$$\left.\left.+\xi_{k,l}^{(i,\,j)}\Delta_{n+1}(k,\,l)\right\}-\left(\alpha\xi_{0,1}^{(i,\,j)}+\gamma\xi_{1,0}^{(i,\,j)}\right)\right]$$
$$\times\left[\sum_{f=0}^{t}\sum_{h=0}^{s}\left\{p_{hf}\left(\delta_{hf}\xi_{h,f}^{(s,t)}-c_1(h+\lambda)\xi_{h+1,f}^{(s,t)}-c_2(f+\mu)\xi_{h,f+1}^{(i,\,j)}\right)\right.\right.$$
$$\left.\left.\left.+\xi_{h,f}^{(s,t)}\Delta_{n+1}(h,\,f)\right\}-\left(\alpha\xi_{0,1}^{(s,t)}+\gamma\xi_{1,0}^{(s,t)}\right)\right]\middle|\mathcal{F}_n\right\},$$

and evaluate the product as four terms, then the above is equivalent to

$$\left[\sum_{l=0}^{j}\sum_{k=0}^{i}p_{kl}\left(\delta_{kl}\xi_{k,l}^{(i,\,j)}-c_1(k+\lambda)\xi_{k+1,l}^{(i,\,j)}-c_2(l+\mu)\xi_{k,l+1}^{(i,\,j)}\right)-(\alpha\xi_{0,1}^{(i,\,j)}+\gamma\xi_{1,0}^{(i,\,j)})\right]$$
$$\times\left[\sum_{f=0}^{t}\sum_{h=0}^{s}p_{hf}\left(\delta_{hf}\xi_{h,f}^{(s,t)}-c_1(h+\lambda)\xi_{h+1,f}^{(s,t)}-c_2(f+\mu)\xi_{h,f+1}^{(i,\,j)}\right)\right.$$

$$
\begin{aligned}
&- \left(\alpha\xi_{0,1}^{(s,t)} + \gamma\xi_{1,0}^{(s,t)}\right)\Bigg] \\
&+ \Bigg[\sum_{l=0}^{j}\sum_{k=0}^{i} p_{kl}\left(\delta_{kl}\xi_{k,l}^{(i,j)} - c_1(k+\lambda)\xi_{k+1,l}^{(i,j)} - c_2(l+\mu)\xi_{k,l+1}^{(i,j)}\right) \\
&- \left(\alpha\xi_{0,1}^{(i,j)} + \gamma\xi_{1,0}^{(i,j)}\right)\Bigg] \\
&\times \left(\sum_{f=0}^{t}\sum_{h=0}^{s}\xi_{h,f}^{(s,t)} p_{hf}\right) \\
&+ \Bigg[\sum_{f=0}^{t}\sum_{h=0}^{s} p_{hf}\left(\delta_{hf}\xi_{h,f}^{(s,t)} - c_1(h+\lambda)\xi_{h+1,f}^{(s,t)} - c_2(f+\mu)\xi_{h,f+1}^{(i,j)}\right) \\
&- \left(\alpha\xi_{0,1}^{(s,t)} + \gamma\xi_{1,0}^{(s,t)}\right)\Bigg] \\
&\times \left(\sum_{l=0}^{j}\sum_{k=0}^{i}\xi_{k,l}^{(i,j)} p_{kl}\right) \\
&+ \mathbf{E}\left[\left(\sum_{l=0}^{j}\sum_{k=0}^{i}\xi_{k,l}^{(i,j)}\Delta_{n+1}(k,l)\right)\left(\sum_{f=0}^{t}\sum_{h=0}^{s}\xi_{h,f}^{(s,t)}\Delta_{n+1}(h,f)\right)\Bigg|\mathcal{F}_n\right] \\
=:\ & A(i,j,s,t) \\
&+ \mathbf{E}\left[\left(\sum_{l=0}^{j}\sum_{k=0}^{i}\xi_{k,l}^{(i,j)}\Delta_{n+1}(k,l)\right)\left(\sum_{f=0}^{t}\sum_{h=0}^{s}\xi_{h,f}^{(s,t)}\Delta_{n+1}(h,f)\right)\Bigg|\mathcal{F}_n\right] \qquad (4.33) \\
=\ & A(i,j,s,t) + \sum_{(k,l)}\sum_{(h,f)}\xi_{k,l}^{(i,j)}\xi_{h,f}^{(s,t)}\mathbf{E}\left[\Delta_{n+1}(k,l)\Delta_{n+1}(h,f)|\mathcal{F}_n\right].
\end{aligned}
$$

Hence, we need the asymptotic form of the sum

$$
\sum_{(k,l)}\sum_{(h,f)}\xi_{k,l}^{(i,j)}\xi_{h,f}^{(s,t)}\mathbf{E}\left[\Delta_{n+1}(k,l)\Delta_{n+1}(h,f)\middle|\mathcal{F}_n\right], \qquad (4.34)
$$

and we divide the summation in (4.34) into two different cases.

(1) Case I: With probability $c_1(r+\lambda)N_n(r,q)/n$, a new edge from $n+1$ to some existing node $v \in V_n$ with $D_n(v) = (r,q)$ is created and this necessitates

$$
\begin{aligned}
\Delta_{n+1}(r,q) &= -1, \quad \text{since an } (r,q)\text{-node is destroyed,} \\
\Delta_{n+1}(r+1,q) &= 1, \quad \text{since an } (r+1,q)\text{-node is created,} \\
\Delta_{n+1}(0,1) &= 1, \quad \text{since a } (0,1)\text{-node is created.}
\end{aligned}
$$

The other case follows by a similar reasoning:

(2) Case II: With probability $c_2(q+\mu)N_n(r,q)/n$, a new edge from some existing node $v \in V_n$ (with $D_n(v) = (r,q)$) to $n+1$ is created such that

$$\Delta_{n+1}(r,q) = -1, \quad \Delta_{n+1}(r,q+1) = 1, \quad \Delta_{n+1}(1,0) = 1.$$

Take Case I as an example, we see that

$$\begin{aligned}
&\Delta_{n+1}(k,l)\Delta_{n+1}(h,f) \\
&\quad = \begin{cases} 1 & \text{if } ((k,l),(h,f)) \in \{((r,q),(r,q)), ((r{+}1,q),(r{+}1,q)), \\ & \quad ((0,1),(0,1)), ((r{+}1,q),(0,1)), ((0,1),(r{+}1,q))\}; \\ -1 & \text{if } \big((k,l),(h,f)\big) \in \{((r{+}1,q),(r,q)), ((r,q),(0,1)), \\ & \quad ((r,q),(r{+}1,q)), ((0,1),(r,q))\}; \\ 0 & \text{otherwise.} \end{cases}
\end{aligned} \tag{4.35}$$

Let $E_1^{(r,q)}$ denote the event described in Case I where node $n+1$ attaches to $v \in V_n$ with $D_n(v) = (r,q)$. Then, on the event $\mathcal{E}_1 := \bigcup_{(r,q)} E_1^{(r,q)}$, (4.35) gives that asymptotically,

$$\begin{aligned}
&\sum_{(k,l)}\sum_{(h,f)} \xi_{k,l}^{(i,j)}\xi_{h,f}^{(s,t)}\mathbf{E}\left[\Delta_{n+1}(k,l)\Delta_{n+1}(h,f)1_{\mathcal{E}_1}\middle|\mathcal{F}_n\right] \\
&\quad = \sum_{(r,q)} \mathbf{P}(E_1^{(r,q)})\left(\xi_{r+1,q}^{(i,j)} + \xi_{01}^{(i,j)} - \xi_{r,q}^{(i,j)}\right)\left(\xi_{r+1,q}^{(s,t)} + \xi_{0,1}^{(s,t)} - \xi_{r,q}^{(s,t)}\right) \\
&\quad = \sum_{q=0}^{n}\sum_{r=0}^{n} c_1(r+\lambda)\frac{N_n(r,q)}{n}\left(\xi_{r+1,q}^{(i,j)} + \xi_{0,1}^{(i,j)} - \xi_{r,q}^{(i,j)}\right) \\
&\qquad \times \left(\xi_{r+1,q}^{(s,t)} + \xi_{0,1}^{(s,t)} - \xi_{r,q}^{(s,t)}\right).
\end{aligned} \tag{4.36}$$

Define $\mathcal{E}_2$ in the same way with respect to Case II, i.e., $\mathcal{E}_2 := \bigcup_{(r,q)} E_2^{(r,q)}$, with $E_2^{(r,q)}$ being the event described in Case II, where node $n+1$ attaches to $v \in V_n$ with $D_n(v) = (r,q)$. Then, similar calculations to (4.36) give $\sum_{(k,l)}\sum_{(h,f)} \xi_{k,l}^{(i,j)}\xi_{h,f}^{(s,t)}\mathbf{E}\left[\Delta_{n+1}(k,l)\Delta_{n+1}(h,f)\middle|\mathcal{F}_n;\mathcal{E}_i\right]$ for $i=2$. Also, (4.29) and (4.30) show that $\mathbf{E}\left[(\Delta_{n+1}(0,1))^2\middle|\mathcal{F}_n\right]$ and $\mathbf{E}\left[(\Delta_{n+1}(1,0))^2\middle|\mathcal{F}_n\right]$ take different forms from the other cases (cf. (4.31)), so we still need to compensate for this.

Considering the case where $(k,l) = (h,f) = (0,1)$, we have by (4.29)

$$\begin{aligned}
&\xi_{0,1}^{(i,j)}\xi_{0,1}^{(s,t)}\mathbf{E}\left[(\Delta_{n+1}(0,1))^2\middle|\mathcal{F}_n\right] \\
&\quad = \xi_{0,1}^{(i,j)}\xi_{0,1}^{(s,t)}\left(\alpha + \delta_{01}\frac{N_n(0,1)}{n}\right) \\
&\quad = \xi_{0,1}^{(i,j)}\xi_{0,1}^{(s,t)}\left(\alpha + c_1\lambda\frac{N_n(0,1)}{n} + c_2(1+\mu)\frac{N_n(0,1)}{n}\right).
\end{aligned}$$

Note that $\xi_{0,1}^{(i,j)}\xi_{0,1}^{(s,t)}c_2(1+\mu)\frac{N_n(0,1)}{n}$ has been covered while calculating (4.34) with respect to $\mathcal{E}_2$, so we only need to add $\xi_{0,1}^{(i,j)}\xi_{0,1}^{(s,t)}\left(\alpha + c_1\lambda\frac{N_n(0,1)}{n}\right)$ to our computation. Similar arguments also apply to $(k,l)=(h,f)=(1,0)$, but instead we add $\left(\gamma + c_2\mu\frac{N_n(1,0)}{n}\right)\xi_{1,0}^{(i,j)}\xi_{1,0}^{(s,t)}$ for compensation.

Taking all these into account, we get

$$\begin{aligned}
&\sum_{(k,l)}\sum_{(h,f)}\xi_{k,l}^{(i,j)}\xi_{h,f}^{(s,t)}\mathbf{E}\left[\Delta_{n+1}(k,l)\Delta_{n+1}(h,f)\middle|\mathcal{F}_n\right]\\
&\quad=\left(\alpha + c_1\lambda\frac{N_n(0,1)}{n}\right)\xi_{0,1}^{(i,j)}\xi_{0,1}^{(s,t)}+\left(\gamma + c_2\mu\frac{N_n(1,0)}{n}\right)\xi_{1,0}^{(i,j)}\xi_{1,0}^{(s,t)}\\
&\qquad+\sum_{q=0}^{n}\sum_{r=0}^{n}\left\{c_1(r+\lambda)\frac{N_n(r,q)}{n}\left(\xi_{r+1,q}^{(i,j)}+\xi_{0,1}^{(i,j)}-\xi_{r,q}^{(i,j)}\right)\left(\xi_{r+1,q}^{(s,t)}+\xi_{0,1}^{(s,t)}-\xi_{r,q}^{(s,t)}\right)\right.\\
&\qquad\left.+c_2(q+\mu)\frac{N_n(r,q)}{n}\left(\xi_{r,q+1}^{(i,j)}+\xi_{1,0}^{(i,j)}-\xi_{r,q}^{(i,j)}\right)\left(\xi_{r,q+1}^{(s,t)}+\xi_{1,0}^{(s,t)}-\xi_{r,q}^{(s,t)}\right)\right\}.
\end{aligned}$$

Here, we also adopt the convention that $\xi_{k,l}^{(i,j)}=0$ if either $k>i$ or $l>j$, and that $N_n(r,q)=0$ whenever either both r and q are 0 or one of them is $n+1$.

Now applying (3.1) again, we write

$$\begin{aligned}
&\sum_{(k,l)}\sum_{(h,f)}\xi_{k,l}^{(i,j)}\xi_{h,f}^{(s,t)}\mathbf{E}\left[\Delta_{n+1}(k,l)\Delta_{n+1}(h,f)\middle|\mathcal{F}_n\right]\\
&\quad\to\left(\alpha + c_1\lambda p_{01}\right)\xi_{0,1}^{(i,j)}\xi_{0,1}^{(s,t)}+\left(\gamma + c_2\mu p_{10}\right)\xi_{1,0}^{(i,j)}\xi_{1,0}^{(s,t)}\\
&\qquad+\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}\left\{c_1(r+\lambda)p_{rq}\left(\xi_{r+1,q}^{(i,j)}+\xi_{0,1}^{(i,j)}-\xi_{r,q}^{(i,j)}\right)\left(\xi_{r+1,q}^{(s,t)}+\xi_{0,1}^{(s,t)}-\xi_{r,q}^{(s,t)}\right)\right.\\
&\quad\left.+c_2(q+\mu)p_{rq}\left(\xi_{r,q+1}^{(i,j)}+\xi_{1,0}^{(i,j)}-\xi_{r,q}^{(i,j)}\right)\left(\xi_{r,q+1}^{(s,t)}+\xi_{1,0}^{(s,t)}-\xi_{r,q}^{(s,t)}\right)\right\}=:B(i,j,s,t)
\end{aligned}\tag{4.37}$$

a.s. as $n\to\infty$. Putting (4.33) and (4.37) together, we conclude that, with probability 1,

$$\mathbf{E}\left[\left(\frac{M_{n+1}(i,j)-M_n(i,j)}{\prod_{d=i+j-1}^{n}\left(1-\frac{\delta_{ij}}{d}\right)^{-1}}\right)\left(\frac{M_{n+1}(s,t)-M_n(s,t)}{\prod_{d=s+t-1}^{n}\left(1-\frac{\delta_{st}}{d}\right)^{-1}}\right)\middle|\mathcal{F}_n\right]\to C(i,j,s,t),\tag{4.38}$$

where $C(i,j,s,t):=A(i,j,s,t)+B(i,j,s,t)$.

Recall that $b_{i,j,n+1}^{(i,j)}=\prod_{d=1}^{n}(1-\delta_{ij}/d)^{-1}$. By Stirling's formula, as $n\to\infty$

$$\begin{aligned}
b_{i,j,n+1}^{(i,j)}&=\prod_{d=i+j-1}^{n}\frac{d}{d-\delta_{ij}}=\frac{\Gamma(n+1)/\Gamma(i+j-1)}{\Gamma(n+1-\delta_{ij})/\Gamma(i+j-1-\delta_{ij})}\\
&\sim n^{\delta_{ij}}\frac{\Gamma(i+j-1-\delta_{ij})}{\Gamma(i+j-1)},
\end{aligned}\tag{4.39}$$

so that as a function of n, $b^{(i,j)}_{i,j,n+1}$ is regularly varying with index δ_{ij}. Therefore, (4.38) becomes

$$\begin{aligned}
&\mathbf{E}\left[\left(\frac{M_{n+1}(i,j)-M_n(i,j)}{n^{\delta_{ij}}}\right)\left(\frac{M_{n+1}(s,t)-M_n(s,t)}{n^{\delta_{st}}}\right)\middle|\mathcal{F}_n\right] \\
&\quad\longrightarrow C(i,j,s,t)\frac{\Gamma(i+j-1-\delta_{ij})}{\Gamma(i+j-1)}\frac{\Gamma(s+t-1-\delta_{st})}{\Gamma(s+t-1)} \\
&\quad =: \tau(i,j,s,t). \qquad (4.40)
\end{aligned}$$

4.2.7. Applying the martingale central limit theorem

We now have the material necessary to verify the conditions in Proposition 4.1. Fix non-negative integers $I, O \in \{0,1,2,\ldots\}$ and define for $I \vee O + 1 \le m \le n$

$$X_{n,m,i,j} = \frac{M_m(i,j)-M_{m-1}(i,j)}{n^{\delta_{ij}+1/2}}, \quad 0 \le i \le I, 0 \le j \le O,$$

and with (s,t) satisfying $0 \le s \le I,\ 0 \le t \le O$, also define

$$\begin{aligned}
G_{n,m}(i,j,s,t) &:= \mathbf{E}(X_{n,m,i,j}X_{n,m,s,t}|\mathcal{F}_{m-1}) \\
&= n^{-(\delta_{ij}+\delta_{st}+1)}\mathbf{E}[(M_m(i,j) \\
&\qquad -M_{m-1}(i,j))(M_m(s,t)-M_{m-1}(s,t))|\mathcal{F}_{m-1}].
\end{aligned}$$

We know from (4.40) that

$$nG_{n,n}(i,j,s,t) \to \tau(i,j,s,t), \quad \text{as } n \to \infty, \qquad (4.41)$$

and that

$$G_{n,m}(i,j,s,t) = \frac{m^{\delta_{ij}+\delta_{st}}}{n^{1+\delta_{ij}+\delta_{st}}} m G_{m,m}(i,j,s,t).$$

Hence, by Karamata's theorem on integration of regularly varying functions, using (4.41), we have

$$\begin{aligned}
V_n(i,j,s,t) := \sum_{m=I\vee O+1}^{n} G_{n,m}(i,j,s,t) &= \frac{\sum_{m=I\vee O}^{n} m^{1+\delta_{ij}+\delta_{st}} G_{m,m}(i,j,s,t)}{n^{1+\delta_{ij}+\delta_{st}}} \\
&\sim \frac{n\cdot n^{1+\delta_{ij}+\delta_{st}} G_{n,n}(i,j,s,t)}{(1+\delta_{ij}+\delta_{st})n^{1+\delta_{ij}+\delta_{st}}} \sim \frac{\tau(i,j,s,t)}{1+\delta_{ij}+\delta_{st}} = \sigma^2(i,j,s,t). \qquad (4.42)
\end{aligned}$$

So the $\big((I+1)\times(O+1)\times(I+1)\times(O+1)\big)$ dimensional matrix converges

$$\Big(V_n(i,j,s,t); 0 \le i,s \le I,\ 0 \le j,t \le O\Big) \to \Sigma_{IO} = (\sigma^2(i,j,s,t)), \qquad (4.43)$$

as required by Propositon 4.1. For each pair (i,j) such that $0 \le i \le I,\ 0 \le j \le O$,

from the definition of $M_n(i, j)$ in (4.6)

$$\frac{M_n(i, j)}{n^{\delta_{ij}+1/2}} = \sum_{l=0}^{j}\sum_{k=0}^{i} \frac{b_{k,l,n}^{(i,j)}}{n^{\delta_{ij}}}\left(\frac{N_n(k, l) - \nu_n(k, l)}{\sqrt{n}}\right)$$
$$\sim \frac{\Gamma(i + j - 1 - \delta_{ij})}{\Gamma(i + j - 1)} \sum_{l=0}^{j}\sum_{k=0}^{i} \xi_{k,l}^{(i,j)}\left(\frac{N_n(k, l) - \nu_n(k, l)}{\sqrt{n}}\right)$$

and this lets us write the matrix equation (with $o_p(1)$ terms dropped)

$$\left(\frac{M_n(i, j)}{n^{\delta_{ij}+1/2}}; 0 \le i \le I, 0 \le j \le O\right)$$
$$=: K_{IO}\left(\frac{N_n(k, l) - \nu_n(k, l)}{\sqrt{n}} : 0 \le k \le I,\ 0 \le l \le O\right)^T, \tag{4.44}$$

where we think of $\big((N_n(k, l) - \nu_n(k, l)/\sqrt{n}, 0 \le k \le I,\ 0 \le l \le O\big)$ as a $(I + 1) \times (O + 1)$ dimensional column vector. Relation (4.44) results from the definitions of $\xi_{k,l}^{(i,j)}$

$$\lim_{n\to\infty} b_{k,l,n}^{(i,j)}/b_{i,j,n}^{(i,j)} \to \xi_{k,l}^{(i,j)} \quad \text{as } n \to \infty,$$

provided that we set $b_{k,l,n}^{(i,j)} = 0$ if either $k = i + 1$ or $l = j + 1$. Then, similar to (4.39)

$$b_{i,j,n}^{(i,j)} = \prod_{d=i+j-1}^{n-1} \frac{d}{d - \delta_{ij}} = \frac{\Gamma(n)/\Gamma(i + j - 1)}{\Gamma(n - \delta_{ij})/\Gamma(i + j - 1 - \delta_{ij})} \sim n^{\delta_{ij}} \frac{\Gamma(i + j - 1 - \delta_{ij})}{\Gamma(i + j - 1)},$$

thus giving the equivalence relationship in (4.44).

In order to apply Proposition 4.1 to conclude (4.5), we must verify conditions (i) and (ii) of the Proposition. Condition (i) of Proposition 4.1 is already satisfied by (4.42), so we just need to consider condition (ii). Since by (4.29)–(4.31) the differences are bounded, i.e.,

$$|(N_n(i, j) - \nu_n(i, j)) - (N_{n-1}(i, j) - \nu_{n-1}(i, j))| \le 2 \text{ for all } (i, j),$$

then we claim that for n large enough, the events $\{|X_{n,m,i,j}| > \varepsilon\}$ become the empty set for all $m \le n$ and all (i, j). This can be observed from the following. For some constant κ_{ij},

$$\{|X_{n,m,i,j}| > \varepsilon\} = \{|M_m(i, j) - M_{m-1}(i, j)| > \varepsilon n^{\delta_{ij}+1/2}\}$$
$$\subseteq \{|\kappa_{ij}| m^{\delta_{ij}} > \varepsilon n^{\delta_{ij}+1/2}\}$$
$$\subseteq \{|\kappa_{ij}| n^{\delta_{ij}} > \varepsilon n^{\delta_{ij}+1/2}\},$$

which becomes the empty set for large n. Therefore, as $n \to \infty$

$$\mathbf{P}(\mathbf{1}_{\{|X_{n,m,i,j}|>\varepsilon\}} \to 0) \to 1.$$

This verifies the second condition.

Recall that calculations in (4.42) and (4.43) gives the covariance matrix Σ_{IO}. Applying Proposition 4.1 yields

$$K_{IO}\left(\frac{N_n(i,j)-\nu_n(i,j)}{\sqrt{n}} : 0 \le i \le I,\ 0 \le j \le O\right)^T \Rightarrow N(0, \Sigma_{I,O}) \tag{4.45}$$

in $\mathbb{R}^{(I+1)(O+1)}$. If we assume further that K_{IO} is invertible, then the convergence in (4.45) can be rewritten as

$$\left(\frac{N_n(i,j)-\nu_n(i,j)}{\sqrt{n}} : 0 \le i \le I,\ 0 \le j \le O\right) \Rightarrow N(0, K_{IO}^{-1}\Sigma_{I,O}K_{IO}^{-T}).$$

Applying Lemma 3.1, we then see that (3.6) holds for fixed I and O.

To avoid non-degenerate limits, we need to make sure that the asymptotic variances given in matrix $\Sigma_{I,O}$ are positive for fixed I and O. It suffices to check that for $0 \le i \le I, 0 \le j \le O$,

$$\lim_{n\to\infty} \frac{\mathcal{V}_n(i,j)}{n} := \lim_{n\to\infty} \frac{\mathrm{Var}(N_n(i,j))}{n} > 0. \tag{4.46}$$

From the definition,

$$\begin{aligned}\mathcal{V}_{n+1}(i,j) &= \mathbf{E}\left[\left(N_{n+1}(i,j)\right)^2\right] - \left(\nu_{n+1}(i,j)\right)^2 \\ &= \mathbf{E}\left[\mathbf{E}\left((N_n(i,j)+\Delta_{n+1}(i,j))^2 \middle| \mathcal{F}_n\right)\right] - \left(\nu_{n+1}(i,j)\right)^2.\end{aligned}$$

For $(i,j) \notin \{(0,1),(1,0)\}$, we have from (4.31),

$$\begin{aligned}&\mathbf{E}\left((N_n(i,j)+\Delta_{n+1}(i,j))^2 \middle| \mathcal{F}_n\right) \\ &= (N_n(i,j))^2 + 2N_n(i,j)\mathbf{E}(\Delta_{n+1}(i,j)|\mathcal{F}_n) + \mathbf{E}\left((\Delta_{n+1}(i,j))^2 \middle| \mathcal{F}_n\right) \\ &= (N_n(i,j))^2 + 2N_n(i,j) \\ &\quad \times \left(c_1(i-1+\lambda)\frac{N_n(i-1,j)}{n} + c_2(j-1+\mu)\frac{N_n(i,j-1)}{n} - \delta_{ij}\frac{N_n(i,j)}{n}\right) \\ &\quad + c_1(i-1+\lambda)\frac{N_n(i-1,j)}{n} + c_2(j-1+\mu)\frac{N_n(i,j-1)}{n} + \delta_{ij}\frac{N_n(i,j)}{n},\end{aligned} \tag{4.47}$$

and using (4.2) gives

$$\begin{aligned}&(\nu_{n+1}(i,j))^2 \\ &= \left(\nu_n(i,j) + c_1(i-1+\lambda)\frac{\nu_n(i-1,j)}{n}\right. \\ &\quad \left. + c_2(j-1+\mu)\frac{\nu_n(i,j-1)}{n} - \delta_{ij}\frac{\nu_n(i,j)}{n}\right)^2 \\ &= (\nu_n(i,j))^2 + 2\nu_n(i,j)\left[c_1(i-1+\lambda)\frac{\nu_n(i-1,j)}{n}\right.\end{aligned}$$

$$+c_2(j-1+\mu)\frac{\nu_n(i,\,j-1)}{n}-\delta_{ij}\frac{\nu_n(i,\,j)}{n}\Bigg]$$
$$+\left[c_1(i-1+\lambda)\frac{\nu_n(i-1,\,j)}{n}+c_2(j-1+\mu)\frac{\nu_n(i,\,j-1)}{n}-\delta_{ij}\frac{\nu_n(i,\,j)}{n}\right]^2. \tag{4.48}$$

Therefore, taking the expectation on both sides of (4.47) and subtracting (4.48) from it give

$$\begin{aligned}\mathcal{V}_{n+1}(i,\,j)=\mathcal{V}_n(i,\,j)\left(1-\frac{2\delta_{ij}}{n}\right)&+\frac{2c_1(i-1+\lambda)}{n}\mathbf{E}[N_n(i,\,j)N_n(i-1,\,j)\\&-\nu_n(i,\,j)\nu_n(i-1,\,j)]\\&+\frac{2c_2(j-1+\mu)}{n}\mathbf{E}[N_n(i,\,j)N_n(i,\,j-1)-\nu_n(i,\,j)\nu_n(i,\,j-1)]\\&+R_{n+1}(i,\,j),\end{aligned}$$

where as $n\to\infty$,

$$\begin{aligned}R_{n+1}(i,\,j)\to\;&c_1(i-1+\lambda)p_{i-1,j}+c_2(j-1+\mu)p_{i,j-1}+\delta_{ij}p_{ij}\\&-[c_1(i-1+\lambda)p_{i-1,j}+c_2(j-1+\mu)p_{i,j-1}-\delta_{ij}p_{ij}]^2\\=\;&(1+2\delta_{ij})p_{ij}-p_{ij}^2\\=\;&2\delta_{ij}p_{ij}+p_{ij}(1-p_{ij})>0,\end{aligned}$$

since $p_{ij}\in(0,\,1]$ and $\delta_{ij}>0$. Note that here $p_{ij}\neq 0$ for all $(i,\,j)$: the recursion in (3.4) shows that both $p_{01},\,p_{10}>0$ as we assume $\alpha,\,\gamma>0$; it also follows that $p_{ij}=0$ for all $(i,\,j)\notin\{(0,\,1),\,(1,\,0)\}$ if we assume $p_{ij}=0$ for some $(i,\,j)$, which is impossible since we initiate the graph with a single node v and $D_1(v)=(1,\,1)$.

Let L_{ij} denote the limit of $R_{n+1}(i,\,j)$, then there exists n_0 such that for all $n\geq n_0$, $R_n(i,\,j)\geq\frac{1}{2}L_{ij}$. Also,

$$\begin{aligned}&\mathbf{E}[N_n(i,\,j)N_n(i-1,\,j)-\nu_n(i,\,j)\nu_n(i-1,\,j)]\\&\quad=\operatorname{cov}(N_n(i,\,j),N_n(i-1,\,j))\geq-(V_n(i,\,j))^{1/2}(V_n(i-1,\,j))^{1/2},\end{aligned}$$

and similarly

$$\mathbf{E}[N_n(i,\,j)N_n(i,\,j-1)-\nu_n(i,\,j)\nu_n(i,\,j-1)]\geq-(V_n(i,\,j))^{1/2}(V_n(i,\,j-1))^{1/2}.$$

We now prove (4.46) by induction. The base case when $n=1$ is trivial. For $n\geq 2$, suppose that $\mathcal{V}_n(i-1,\,j)\geq a_{i-1,j}n$ and $\mathcal{V}_n(i,\,j-1)\geq a_{i,j-1}n$ for some $a_{i-1,j},\,a_{i,j-1}>0$, then for all $n\geq n_0$,

$$\begin{aligned}\mathcal{V}_{n+1}(i,\,j)\geq\mathcal{V}_n(i,\,j)\left(1-\frac{\delta_{ij}}{n}\right)&-2c_1(i-1+\lambda)\left(\frac{\mathcal{V}_n(i,\,j)}{n}\right)^{1/2}\left(\frac{\mathcal{V}_n(i-1,\,j)}{n}\right)^{1/2}\\&-2c_2(j-1+\mu)\left(\frac{\mathcal{V}_n(i,\,j)}{n}\right)^{1/2}\left(\frac{\mathcal{V}_n(i,\,j-1)}{n}\right)^{1/2}+\frac{1}{2}L_{ij}.\end{aligned}$$

We therefore conclude that

$$\mathcal{V}_{n+1}(i,\,j) \geq \mathcal{V}_n(i,\,j)\left(1-\frac{K_1^{(i,j)}}{n}\right) - K_2^{(i,j)}\left(\frac{\mathcal{V}_n(i,\,j)}{n}\right)^{1/2} + \frac{1}{2}L_{ij}, \quad \forall n \geq n_0,$$

where $K_1^{(i,j)}, K_2^{(i,j)} > 0$ are positive constants.

If $\frac{\mathcal{V}_n(i,j)}{n} \leq \left(\frac{L_{ij}}{4K_2^{(i,j)}}\right)^2$, then

$$-K_2^{(i,j)}\left(\frac{\mathcal{V}_n(i,\,j)}{n}\right)^{1/2} \geq \frac{1}{4}L_{ij}.$$

If $\frac{\mathcal{V}_n(i,j)}{n} \geq \left(\frac{L_{ij}}{4K_2^{(i,j)}}\right)^2$, then

$$\left(\frac{\mathcal{V}_n(i,\,j)}{n}\right)^{1/2} \leq \frac{4K_2^{(i,j)}}{L_{ij}} \cdot \frac{\mathcal{V}_n(i,\,j)}{n}.$$

In either case, for all $n \geq n_0$ we still have

$$\begin{aligned}\mathcal{V}_{n+1}(i,\,j) &\geq \mathcal{V}_n(i,\,j)\left(1-\frac{K_1^{(i,j)}}{n}\right) - K_2^{(i,j)}\frac{4K_2^{(i,j)}}{L_{ij}} \cdot \frac{\mathcal{V}_n(i,\,j)}{n} + \frac{1}{4}L_{ij} \\ &= \mathcal{V}_n(i,\,j)\left(1-\frac{K_1^{(i,j)} + 4\left(K_2^{(i,j)}\right)^2/L_{ij}}{n}\right) + \frac{1}{4}L_{ij}.\end{aligned}$$

Since $K^{(i,j)} := K_1^{(i,j)} + 4\left(K_2^{(i,j)}\right)^2/L_{ij} > 0$, then for $n \geq n_0$,

$$\begin{aligned}\mathcal{V}_{n+1}(i,\,j) &\geq \frac{1}{4}L_{ij}\left[1+\left(1-\frac{K^{(i,j)}}{n}\right)+\left(1-\frac{K^{(i,j)}}{n}\right)^2+\cdots+\left(1-\frac{K^{(i,j)}}{n}\right)^{n-n_0}\right] \\ &= \frac{1}{4}L_{ij}\frac{1-(1-K^{(i,j)}/n)^{n-n_0+1}}{K^{(i,j)}/n} \\ &\sim \frac{L_{ij}}{4K^{(i,j)}}(1-e^{-K^{(i,j)}})n > 0, \qquad \text{as } n \to \infty.\end{aligned}$$

So we are done with proving (4.46), thus completing the proof for Theorem 3.1.

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