# Ratios of Ordered Points of Point Processes with Regularly Varying Intensity Measures 

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#### Abstract

We study limiting properties of ratios of ordered points of point processes whose intensity measures have regularly varying tails, giving a systematic treatment which points the way to "large-trimming" properties of extremal processes and a variety of applications. Our point process approach facilitates a connection with the negative binomial process of Gregoire (II.984) and consequently to certain generalised versions of the Poisson-Dirichlet distribution.


## 1 Introduction

Recent work on ratios of ordered Poisson points and ordered jumps of stable subordinators and other Lévy processes due to Kevei \& Mason (2014) and the present authors in Buchmann, Fan \& Maller (2016), Ipsen \& Maller (2017a) and Buchmann, Maller \& Resnick (2016) placed an emphasis on limiting properties of those ratios, and on "trimmed" versions of the process generating the points, which may have been a subordinator or a more general Lévy process.

Our aim in this paper is to give a systematic treatment of the limiting behaviour of ratios of ordered Poisson points. As is natural, we take a point process approach and make special connection with the negative binomial process whose relevance in the present context was brought out in 【psen \& Maller (2017b). This connection via ratios of points enabled the construction of a generalised kind of Poisson-Dirichlet distribution which can be added to the repertoire of available models for data analytic purposes.

[^0]A related topic is the behaviour of two dimensional Poisson points ordered by the second component when the $r$ highest points are deleted. Such processes were explored in Buchmann, Maller \& Resnick (2016), and the present results provide impetus for further investigations of this kind.

The paper is structured as follows. In Section we set up the point processes to be studied in Section [3, notably a Poisson point process $\mathbb{D}$ on $\mathbb{R}^{+}=(0, \infty)$, and subsidiary point processes $\mathbb{D}_{t}^{(n)}$ and $\mathbb{D}_{t}^{(r, r+n)}$ consisting of ratios of the ordered points in $\mathbb{D}$, where the ordering is by magnitude up till a given time $t>0$, and the normalisation is by the $n$th largest point.

The tail of the canonical measure for the points is assumed to be regularly varying of index $-\alpha, \alpha>0$, at 0 . Under this assumption, Theorem 3.] in Section 3] proves the weak convergence of $\mathbb{D}_{t}^{(r, r+n)}$, as $t \downarrow 0$, to a limit comprised of a sum of independent point processes on $(0, \infty)$. The first component of the sum represents the joint limiting distribution of ratios larger than 1 of points in $\mathbb{D}$, conveniently expressed as the distribution of the order statistics of certain i.i.d. (independent, identically distributed) random variables (rvs); and the second component is a negative binomial point process, representing the limiting distribution of the (infinitely many) ratios smaller than 1.

Further, in Section G, we mention some interesting corollaries of Theorem B.D, stated as separate propositions, and in Section 5 prove a converse result (Theorem 5.1) to the effect that convergence in distribution of ratios (larger or smaller than 1) implies regular variation of the tail of the canonical measure for points in $\mathbb{D}$. We conclude in Section [6 with some history relating to antecedents of these results in the literature of order statistics of i.i.d. rvs, which can be used to suggest further explorations in that area.

## 2 Poisson Point Processes and Ratios of Ordered Points

In this section we set up the point process framework we will use. Let $\mathbb{N}=\{1,2, \ldots\}$ and $\mathbb{N}_{0}=\{0,1,2, \ldots\}$. Suppose $\Pi$ is a Borel measure on $(0, \infty)$, locally finite at infinity. The measure $\Pi$ has finite-valued tail function $\bar{\Pi}:(0, \infty) \rightarrow(0, \infty)$, defined by

$$
\bar{\Pi}(x):=\Pi\{(x, \infty)\}, x>0
$$

a right-continuous, non-increasing function. Assume throughout that $\Pi\{(0, \infty)\}=$ $\bar{\Pi}(0+)=\infty$, so there are infinitely many non-zero points of $\mathbb{D}$ in any right neighborhood of 0 . Let

$$
\bar{\Pi}^{\leftarrow}(x)=\inf \{y>0: \bar{\Pi}(y) \leq x\}, x>0
$$

be the right-continuous inverse of $\bar{\Pi}$. With $\delta_{x}$ denoting a point mass at $x \in(0, \infty)$, let

$$
\begin{equation*}
\mathbb{D}_{t}=\sum_{j>0} \delta_{\Delta_{t}^{(j)}} \tag{2.1}
\end{equation*}
$$

a Poisson point process on $(0, \infty)$ with intensity measure $t \Pi(\mathrm{~d} x)$, where the points are written in decreasing order, possibly with ties,

$$
\infty>\Delta_{t}^{(1)} \geq \cdots \geq \Delta_{t}^{(r)} \geq \cdots>0
$$

A representation detailed in Buchmann, Fan \& Maller (2016) shows how to construct all $\mathbb{D}_{t}$ processes on the same space. Since $\bar{\Pi}(0+)=\infty$, all $\Delta_{t}^{(r)}$ are positive and $\Delta_{t}^{(r)} \downarrow 0$ a.s. as $t \downarrow 0$ for $r \in \mathbb{N}$. Let $\left(\mathfrak{E}_{i}\right)$ be an i.i.d. sequence of exponentially distributed random variables with common parameter $\mathbb{E}_{i}=1$. Then $\Gamma_{r}:=\sum_{i=1}^{r} \mathfrak{E}_{i}$ is a $\operatorname{Gamma}(r, 1)$ random variable, $r \in \mathbb{N}$, and $\left\{\Gamma_{r}, r \geq 1\right\}$ are the points of a homogeneous, unit rate Poisson process on $\mathbb{R}^{+}$. The representation is

$$
\begin{equation*}
\left\{\Delta_{t}^{(i)}\right\}_{i \geq 1} \stackrel{\mathrm{D}}{=}\left\{\bar{\Pi}^{\leftarrow}\left(\Gamma_{i} / t\right)\right\}_{i \geq 1}, t>0 \tag{2.2}
\end{equation*}
$$

For earlier and related representations consult LePage ([1980, [1981); LePage, Woodroofe \& Zinn ([198]); Samorodnitsky \& Taqqu ([9994), p. 21, 30; Resnick ([1987), Ex. 3.38, p.139; Resnick ([1986), Sect. 2.4; and Ferguson \& Klass ([1972).

Write

$$
\mathrm{P}\left(\Gamma_{r} \in \mathrm{~d} x\right)=\frac{x^{r-1} e^{-x} \mathrm{~d} x}{\Gamma(r)} \mathbf{1}_{\{x>0\}}, r \in \mathbb{N},
$$

for the density of $\Gamma_{r}$, which should not be confused with the Gamma function, $\Gamma(r)=\int_{0}^{\infty} x^{r-1} e^{-x} \mathrm{~d} x, r>0$. A beta random variable $\mathrm{B}_{a, b}$ on $(0,1)$ with parameters $a, b>0$ has density function

$$
f_{B}(x)=\frac{\Gamma(a+b)}{\Gamma(a) \Gamma(b)} x^{a-1}(1-x)^{b-1}=\frac{1}{B(a, b)} x^{a-1}(1-x)^{b-1}, 0<x<1 .
$$

Thus

$$
\begin{equation*}
\mathrm{P}\left(\mathrm{~B}_{a, b} \leq x\right)=\frac{1}{B(a, b)} \int_{0}^{x} y^{a-1}(1-y)^{b-1} \mathrm{~d} y:=B(a, b ; x), 0<x<1, \tag{2.3}
\end{equation*}
$$

where $B(a, b ; x)$ is the incomplete Beta function.
We are interested in the convergence behaviour of ratios of the order statistics $\Delta_{t}^{(r)}$, as $t \downarrow 0$. The basic assumption is the regular variation of the tail function $\bar{\Pi}(x)$. Write $R V_{0}(\beta)$ (resp. $R V_{\infty}(\beta)$ ) for the real-valued functions regularly varying at 0 (resp, infinity) with index $\beta$. We have $\bar{\Pi}(x) \in R V_{0 / \infty}(-\alpha), 0 \leq \alpha \leq \infty$, iff

$$
\lim _{x \rightarrow 0 / \infty} \frac{\bar{\Pi}(x \lambda)}{\bar{\Pi}(x)}=\lambda^{-\alpha}, \text { for } \lambda>0
$$

Interpret $\lambda^{-\infty}=0 . \mathbf{1}_{\{\lambda>1\}}+1 . \mathbf{1}_{\{\lambda=1\}}+\infty . \mathbf{1}_{\{\lambda<1\}}$ and $1 / 0 \equiv \infty$. From Bingham, Goldie \& Teugels (19877, p.28-29) we know that $\bar{\Pi}(x) \in R V_{0 / \infty}(-\alpha)$ iff $\bar{\Pi}^{\leftarrow}(x) \in$ $R V_{\infty / 0}(-1 / \alpha)$. The slowly varying functions at 0 or $\infty$ are denoted $R V_{0 / \infty}(0)$ and $R V_{0 / \infty}(\infty)$ are the rapidly varying functions at 0 or $\infty$.

When $\bar{\Pi}(\cdot) \in R V_{0}(-\alpha)$ with $0 \leq \alpha \leq \infty$ or, equivalently, $\bar{\Pi}{ }^{\leftarrow}(\cdot) \in R V_{\infty}(-1 / \alpha)$, we have the easily verified convergence (with the interpretation as above when $\alpha=0$ or $\alpha=\infty$ )

$$
\begin{equation*}
t \overline{\bar{\Pi}}\left(u \bar{\Pi}^{\leftarrow}(y / t)\right) \sim \frac{\bar{\Pi}\left(u y^{-1 / \alpha} \bar{\Pi}^{\leftarrow}(1 / t)\right)}{\bar{\Pi}\left(\bar{\Pi}^{\leftarrow}(1 / t)\right)} \rightarrow u^{-\alpha} y \text { as } t \downarrow 0, \text { for all } u, y>0 \tag{2.4}
\end{equation*}
$$

## 3 Ratios of Ordered Points

In this section we give a general result for the point processes of ratios of ordered points of $\mathbb{D}_{t}$. Fix $r \in \mathbb{N}_{0}, n \in \mathbb{N}$ and $t>0$. Define the point processes on $(0, \infty)$ :

$$
\begin{equation*}
\mathbb{D}_{t}^{(n)}:=\sum_{i \geq n+1} \delta_{\left\{\Delta_{t}^{(i)} / \Delta_{t}^{(n)}\right\}} \tag{3.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathbb{D}_{t}^{(r, r+n)}:=\sum_{i \geq r+1} \delta_{\left\{\Delta_{t}^{(i)} / \Delta_{t}^{(r+n)}\right\}}=\sum_{i=1}^{n-1} \delta_{\left\{\Delta_{t}^{(r+i)} / \Delta_{t}^{(r+n)}\right\}}+1+\mathbb{D}_{t}^{(r+n)} \tag{3.2}
\end{equation*}
$$

Conditionally on $\left\{\Delta_{t}^{(n)}=z\right\}, z>0$, the points $\left(\Delta_{t}^{(i)}\right)_{i \geq n+1}$ comprise a Poisson point process with intensity measure $t \Pi$ restricted to $(0, z)$. Thus, the Laplace functional of $\mathbb{D}_{t}^{(n)}$, conditional on $\left\{\Delta_{t}^{(n)}=z\right\}$, is

$$
\begin{align*}
\mathrm{E}\left(e^{-\mathbb{D}_{t}^{(n)}(f)} \mid \Delta_{t}^{(n)}=z\right) & =\mathrm{E}\left(\exp \left(-\int_{0<x<1} f(x) \mathbb{D}_{t}^{(n)}(\mathrm{d} x)\right) \mid \Delta_{t}^{(n)}=z\right) \\
& =\exp \left(-t \int_{0<x<1}\left(1-e^{-f(x)}\right) \Pi(z \mathrm{~d} x)\right) \tag{3.3}
\end{align*}
$$

where $f \in \mathcal{F}_{+}$, the nonnegative measurable functions on $\mathbb{R}^{+}$.
Let $\mathbb{S}$ be a Poisson point process on $(0, \infty)$ with intensity measure $\Lambda(\mathrm{d} x)=$ $\alpha x^{-\alpha-1} \mathbf{1}_{\{x>0\}}, \alpha>0$, represented as $\mathbb{S}=\sum_{i \geq 1} \delta_{\Gamma_{i}^{-1 / \alpha}}$. When $0<\alpha<2$ we can interpret $\Gamma_{i}^{-1 / \alpha}$ as the $i$ th largest jump of a stable process $\left(S_{t}\right)_{0<t \leq 1}$ with Lévy measure $\Lambda$, but we allow any $\alpha>0$. Analogous to ([.]), define

$$
\begin{equation*}
\mathbb{B}^{(n)}=\sum_{i \geq n+1} \delta_{\left\{\left(\Gamma_{i} / \Gamma_{n}\right)^{-1 / \alpha}\right\}}, n \in \mathbb{N} . \tag{3.4}
\end{equation*}
$$

The point process in (3.4) has Laplace functional at $f$ equal to

$$
\begin{equation*}
\mathrm{E}\left(e^{-\mathbb{B}^{(n)}(f)}\right)=\left(1+\int_{0}^{1}\left(1-e^{-f(x)}\right) \Lambda(\mathrm{d} x)\right)^{-n} \tag{3.5}
\end{equation*}
$$

$\mathbb{B}^{(n)}$ is the negative binomial point process with base measure $\Lambda^{*}(\mathrm{~d} x)=\Lambda(\mathrm{d} x) \mathbf{1}_{0<x<1}$, denoted by $\mathcal{B N}\left(n, \Lambda^{*}\right)$, in the notation of Gregoire (11984).

The next theorem shows the weak convergence (denoted by ' ${ }^{\mathrm{D}}$ ') of $\mathbb{D}_{t}^{(r, r+n)}$ as $t \downarrow 0$ to a limit comprised of independent components of $\mathbb{B}^{(r+n)}$ and a mixture of beta random variables.

Theorem 3.1. Suppose $\bar{\Pi}(\cdot) \in R V_{0}(-\alpha), 0<\alpha<\infty$, and $f \in \mathcal{F}_{+}$. Fix $n, r \in \mathbb{N}$. Then
(i) In the space of point measures $M_{p}(0, \infty)$ with the vague topology, as $t \downarrow 0$,

$$
\begin{equation*}
\mathbb{D}_{t}^{(r, r+n)} \xrightarrow{\mathrm{D}} \sum_{j=1}^{\infty} \delta_{\left\{\left(\Gamma_{r+j} / \Gamma_{r+n}\right)^{-1 / \alpha}\right\}}=\sum_{j=1}^{n-1} \delta_{\left\{\left(\Gamma_{r+j} / \Gamma_{r+n}\right)^{-1 / \alpha}\right\}}+\delta_{1}+\mathbb{B}^{(r+n)} \tag{3.6}
\end{equation*}
$$

The limit has Laplace functional at $f$ equal to

$$
\begin{equation*}
\mathrm{E}\left(e^{-f\left(J\left(B_{r, n}^{1 / \alpha}\right)\right)}\right)^{n-1} e^{-f(1)} \mathrm{E}\left(e^{-\mathbb{B}^{(r+n)}(f)}\right) \tag{3.7}
\end{equation*}
$$

where, for each $u \in(0,1), J(u)$ has distribution

$$
\begin{equation*}
\mathrm{P}(J(u) \in \mathrm{d} x)=\frac{\Lambda(\mathrm{d} x) \mathbf{1}_{\{1<x<1 / u\}}}{1-u^{\alpha}}, x>0, \tag{3.8}
\end{equation*}
$$

$B_{r, n}$ is a $\operatorname{Beta}(r, n)$ random variable independent of $J(u)$, and the third factor on the right of (3.7) is determined from (3.5).
(ii) For $r=0$,

$$
\begin{equation*}
\lim _{t \downarrow 0} \mathrm{E}\left(e^{-\mathbb{D}_{t}^{(0, n)}(f)}\right)=\mathrm{E}\left(e^{-f(L)}\right)^{n-1} e^{-f(1)} \mathrm{E}\left(e^{-\mathbb{B}^{(n)}(f)}\right) \tag{3.9}
\end{equation*}
$$

where $L$ is a random variable with distribution

$$
\begin{equation*}
\mathrm{P}(L \in \mathrm{~d} x)=\Lambda(\mathrm{d} x) \mathbf{1}_{\{x>1\}} . \tag{3.10}
\end{equation*}
$$

Proof of Theorem 3.1: (i) Using the representation in (2.2) and the fact that $\bar{\Pi} \leftarrow \in R V_{\infty}(-1 / \alpha)$, we immediately get, as $t \downarrow 0$, with almost sure convergence,

$$
\begin{equation*}
\left(\frac{\Delta_{t}^{(r+j)}}{\Delta_{t}^{(r+n)}} ; j \geq 1\right) \stackrel{\mathrm{D}}{=}\left(\frac{\bar{\Pi}^{\leftarrow}\left(\Gamma_{r+j} / t\right)}{\bar{\Pi}^{\leftarrow}\left(\Gamma_{r+n} / t\right)} ; j \geq 1\right) \rightarrow\left(\left(\frac{\Gamma_{r+j}}{\Gamma_{r+n}}\right)^{-1 / \alpha} ; j \geq 1\right) \tag{3.11}
\end{equation*}
$$

for each $r, n \in \mathbb{N}$. By separating the ratios in the limit process into those bigger than 1 , equal to 1 , or smaller than 1 , we get the form in ( $\mathbf{3 . 6}$ ).

The points in the two limit point processes in (3.6) occur in non-overlapping regions, so, conditionally on $\Gamma_{r+n}$, they are independent of each other. In fact,
by the algebraic properties of gamma distributions, i.e., $\Gamma_{r} / \Gamma_{r+n} \stackrel{\text { D }}{=} B_{r, n}$ with $B_{r, n}$ independent of $\Gamma_{r+n}$, these components are also unconditionally independent. Thus the Laplace transform can be given in the product form of (3.7). Next we will derive the Laplace functional for each component separately.

For ratios bigger than 1, we note from properties of a homogeneous Poisson process that, conditionally on $\Gamma_{r} / \Gamma_{r+n}=B_{r, n}=s$,

$$
\frac{\Gamma_{r+1}}{\Gamma_{r+n}}, \ldots, \frac{\Gamma_{r+n-1}}{\Gamma_{r+n}}
$$

are the order statistics of a uniform sample of size $n-1$ on $(s, 1)$ and the unordered sample has representation

$$
s+(1-s) U_{j} ; j=1 \ldots, n-1,
$$

where $U_{1}, \ldots, U_{n-1}$ are iid uniform on $(0,1)$. Thus for $f \in \mathcal{F}_{+}$,

$$
\begin{aligned}
\left.\operatorname{Eexp}\left(\left.-\sum_{j=1}^{n-1} f\left(\left(\frac{\Gamma_{r+j}}{\Gamma_{r+n}}\right)^{-1 / \alpha}\right) \right\rvert\, B_{r, n}=s\right)=\mathrm{E} \exp \left\{-\sum_{j=1}^{n-1} f\left((1-s) U_{j}+s\right)^{-1 / \alpha}\right)\right\} \\
\left.=\left(\int_{0}^{1} \exp \left\{-f((1-s) u+s)^{-1 / \alpha}\right)\right\} d u\right)^{n-1},
\end{aligned}
$$

and setting $y=((1-s) u+s)^{-1 / \alpha}$ gives

$$
\begin{equation*}
=\left(\int_{1}^{s^{-1 / \alpha}} e^{-f(y)} \alpha y^{-\alpha-1} \frac{d y}{1-s}\right)^{n-1} . \tag{3.12}
\end{equation*}
$$

Take expectations in (3.12) to get

$$
\operatorname{Eexp}\left(-\sum_{j=1}^{n-1} f\left(\left(\frac{\Gamma_{r+j}}{\Gamma_{r+n}}\right)^{-1 / \alpha}\right)\right)=\mathrm{E}\left(\int_{1}^{B_{r, n}^{-1 / \alpha}} e^{-f(y)} \alpha y^{-\alpha-1} \frac{d y}{1-B_{r, n}}\right)^{n-1}
$$

Conditional on $B_{r, n}^{1 / \alpha}$, this is the Laplace functional of $n-1$ i.i.d. rvs with the dsitribution in (3.8).

Next we compute the intensity measure of the limit point process with ratios less than 1, that is, the process $\mathbb{D}_{t}^{(r+n)}=\sum_{j \geq n+1} \delta_{\left\{\left(\Gamma_{r+j} / \Gamma_{r+n}\right)^{-1 / \alpha}\right\}}$ in (3.2). Conditionally on $\Gamma_{r+n}$, the process $\sum_{j \geq n+1} \delta_{\left\{\Gamma_{r+j} / \Gamma_{r+n}\right\}}$ is a Poisson process with mean measure $\Gamma_{r+n} \mathrm{~d} x$, where $\mathrm{d} x$ is the Lebesque measure. Then the image measure of $\Gamma_{r+n} \mathrm{~d} x$ under the map $T: x \mapsto x^{-1 / \alpha}$ is $\Gamma_{r+n} \Lambda(\mathrm{~d} x)$. Hence for any nonnegative
measurable function $f$,

$$
\begin{align*}
& \operatorname{Eexp}\left(-\sum_{j \geq n+1} f\left(\left(\frac{\Gamma_{r+j}}{\Gamma_{r+n}}\right)^{-1 / \alpha}\right)\right) \\
& =\int_{y>0} \exp \left(-\int_{0<x<1}\left(1-e^{-f(x)}\right) y \Lambda(\mathrm{~d} x)\right) \mathrm{P}\left(\Gamma_{r+n} \in \mathrm{~d} y\right) \\
& =\left(1+\int_{0}^{1}\left(1-e^{-f(x)}\right) \Lambda(\mathrm{d} x)\right)^{-r-n} . \tag{3.13}
\end{align*}
$$

Referring to (3.5), this is the Laplace transform of a negative binomial point process $\mathcal{B N}\left(r+n, \Lambda(\mathrm{~d} x) \mathbf{1}_{0<x<1}\right)$ at $f$.
(ii) $(r=0)$ The proof of (B.Y) is very similar. The treatment for ratios smaller than or equal to 1 is exactly the same. For points bigger than 1 , no conditioning is necessary. We omit further details.

Remark 3.1. The first component on the RHS of the limit in (3.7) shows that, after deleting the $r$ largest points, the sum $\sum_{i=r+1}^{r+n-1}\left(\Gamma_{i} / \Gamma_{r+n}\right)^{-1 / \alpha}$ has the distribution of a sum of i.i.d. random variables, once we condition on a $B_{r, n}^{1 / \alpha}$ random variable. The third component on the RHS of (3.7) is the negative binomial point process $\mathbb{B}^{(r+n)}$ with base measure $\Lambda$. So we have the nice representation resulting from the decomposition of the original process into parts including ratios smaller than 1 and greater than 1.

Ratios of jumps of stable subordinators also featured prominently in the work of Pitman \& Yorl ([19.97). Much subsequent related research involved Poisson-Dirichlet distributions and their involvement in fragmentation and coalescence problems; see Bertoin (2006) and references therein. An early influential paper was Kingman (1.975). The resulting processes have found wide application in a variety of applied areas ranging from Bayesian statistics to models for species diversity; see for example the list in Pitman \& Yor ([1996, Sect.1).

When $0<\alpha<1, r=0$, similar results were obtained in Lemma 24 of Pitman \& Yor (1997) but without explicit reference being made to the negative binomial point process of Gregoire (1984). Our result allows the bigger range of $\alpha, \alpha>0$, and generalises to point processes with intensity measures whose tails are regularly varying, rather than dealing only with jumps of subordinators. In general, in our scenario, the points of the limiting process may not be summable. As a special case, for example, we deal elsewhere (in 【psen \& Maller (2017a)) with Lévy processes in the domain of attraction of a stable process with index $\alpha \in(1,2)$; compensating the process is then essential.

In the next section we draw out some ramifications of Theorem [3.1].

## 4 Corollaries, Special Cases and Further Results

Theorem [3.D is expressed as convergence of point processes. In this section we express the theorem in a different form in order to facilitate comparisons with earlier results in the literature; we also extend the result to the $\alpha=0$ or $\alpha=+\infty$ cases and consider limits of conditional distributions.

The discussion is again conveniently divided into parts covering ratios smaller than or greater than 1. Define the ratio

$$
\begin{equation*}
W_{r, n}(t):=\frac{\Delta_{t}^{(r+n)}}{\Delta_{t}^{(r)}}, r, n \in \mathbb{N}, t>0 \tag{4.1}
\end{equation*}
$$

Proposition 4.1 (Ratios smaller than 1). Suppose $\bar{\Pi}(\cdot) \in R V_{0}(-\alpha), 0 \leq \alpha \leq \infty$.
(i) Suppose $0<\alpha<\infty$. Then, for each $n \in \mathbb{N}$,

$$
\begin{equation*}
\mathbb{D}_{t}^{(n)} \xrightarrow{\mathrm{D}} \mathbb{B}^{(n)} \text {, as } t \downarrow 0 \text {, } \tag{4.2}
\end{equation*}
$$

where $\mathbb{B}^{(n)}$ is distributed as $\mathcal{B N}(n, \Lambda)$ with Laplace functional as in (B.5).
(ii) As $t \downarrow 0$, for each $r, n \in \mathbb{N}$,

$$
\begin{equation*}
\left(\frac{\Delta_{t}^{(r+1)}}{\Delta_{t}^{(r)}}, \ldots, \frac{\Delta_{t}^{(r+n)}}{\Delta_{t}^{(r+n-1)}}\right) \xrightarrow{\mathrm{D}}\left(Y_{r}, \ldots, Y_{r+n-1}\right), \tag{4.3}
\end{equation*}
$$

where $Y_{k}, k \in \mathbb{N}$, are mutually independent random variables with $\operatorname{Beta}(k \alpha, 1)$ distributions. When $\alpha=0$ or $\alpha=\infty$, (4.3) remains true with each $Y_{k}$ equal to 0 or with each $Y_{k}$ equal to 1, respectively.
(iii) When $0<\alpha<\infty, W_{r, n}(t)$ in (4..1) has limiting distribution as $t \downarrow 0$ that of

$$
\begin{equation*}
\prod_{i=1}^{n} Y_{r+i-1} \stackrel{\mathrm{D}}{=}\left(\frac{\Gamma_{r}}{\Gamma_{r+n}}\right)^{1 / \alpha} \stackrel{\mathrm{D}}{=} B_{r, n}^{1 / \alpha}=: W_{r, n}, \tag{4.4}
\end{equation*}
$$

where $W_{r, n}$ has density

$$
\begin{equation*}
f_{W_{r, n}}(w)=\frac{\left(1-w^{\alpha}\right)^{n-1} \alpha w^{\alpha r-1}}{B(r, n)}, 0<w<1 . \tag{4.5}
\end{equation*}
$$

Proof of Proposition 4.7: (i) The convergence in (4.2) is an immediate consequence of (3..9).
(ii) When $0<\alpha<\infty$ the convergence in (4.3) follows immediately from ([2.1]). When $\alpha=0$ or $\alpha=\infty$, (B. Bl $^{2}$ ) remains true with the appropriate interpretations as outlined in the discussion leading to (2.4).
(iii) Equation (4.4) is implied by ( 4.3 ) and the density in ( 4.5 ) is easily calculated.

Remark 4.1. Treated as ratios of ordered jumps of a subordinator, Kevei \& Mason (2014) proved the case $n=1$ in (4.3), among other results comparing the magnitudes of ordered jumps of a subordinator with the magnitude of the subordinator itself. Proposition 4.0 is a multidimensional version of their Theorem 1.2, with the $\left(\Delta_{t}\right)$ treated as points in $\mathbb{D}$, in their own right. (They also proved converse results; see Section [5.)

Proposition 4.2 (Ratios greater than 1). Suppose $\bar{\Pi}(\cdot) \in R V_{0}(-\alpha)$ with $0 \leq \alpha \leq$ $\infty$. Take $x_{k} \geq 1$ for $0 \leq k \leq n-1, n=2,3, \ldots, r \in \mathbb{N}$ and $z>0$.
(a) Assume $0<\alpha<\infty$ and $z>0$.
(i) Then, for $0<u<1$,

$$
\begin{align*}
& \lim _{t \downarrow 0} \mathrm{P}\left(\frac{\Delta_{t}^{(r+k)}}{\Delta_{t}^{(r+n)}}>x_{k}, 0 \leq k \leq n-1 \mid W_{r, n}(t)=u, \Delta_{t}^{(r+n)}=\bar{\Pi}^{\leftarrow}(z / t)\right) \\
& =\lim _{t \downarrow 0} \mathrm{P}\left(\frac{\Delta_{t}^{(r+k)}}{\Delta_{t}^{(r+n)}}>x_{k}, 0 \leq k \leq n-1 \mid W_{r, n}(t)=u\right) \\
& =\mathbf{1}_{\left\{u<x_{0}^{-1}\right\}} \mathrm{P}\left(J_{n-1}^{(k)}(u)>x_{k}, 1 \leq k \leq n-1\right), \tag{4.6}
\end{align*}
$$

where $J_{n-1}^{(1)}(u) \geq J_{n-1}^{(2)}(u) \ldots \geq J_{n-1}^{(n-1)}(u)$ are distributed like the decreasing order statistics of $n-1$ independent and identically distributed random variables $\left(J_{i}(u)\right)_{1 \leq i \leq n-1}$, each having the distribution in ([.8).
(ii) For $n, r, x_{k}, z$ as specified,

$$
\begin{align*}
& \lim _{t \downarrow 0} \mathrm{P}\left(\frac{\Delta_{t}^{(r+k)}}{\Delta_{t}^{(r+n)}}>x_{k}, 0 \leq k \leq n-1 \mid \Delta_{t}^{(r+n)}=\bar{\Pi}^{\leftarrow}(z / t)\right) \\
& =\mathrm{P}\left(J_{n-1}^{(k)}\left(\mathrm{B}_{r, n}^{1 / \alpha}\right)>x_{k}, 1 \leq k \leq n-1, \mathrm{~B}_{r, n}^{1 / \alpha} \leq x_{0}^{-1}\right) \tag{4.7}
\end{align*}
$$

where the $J_{i}(u)$ are as in (4.6) and $B_{r, n}$ is a $\operatorname{Beta}(r, n)$ random variable independent of $\left(J_{i}(u)\right)_{1 \leq i \leq n-1}$.
(b) When $\alpha=0$, each ratio $\Delta_{t}^{(r+k)} / \Delta_{t}^{(r+n)} \xrightarrow{\mathrm{P}} \infty$ as $t \downarrow 0$, for $1 \leq k \leq n-1$. When $\alpha=\infty$, each ratio $\Delta_{t}^{(r+k)} / \Delta_{t}^{(r+n)} \xrightarrow{\mathrm{P}} 1$ as $t \downarrow 0$, for $1 \leq k \leq n-1$.

Proof of Proposition 4.2: Equations (4.6) and (4.7) are implicitly proved in the proof of Theorem [.D. Part (b) follows from similar arguments as in Part (ii) of Proposition 4.1.

Remark 4.2. (i) In Part (a)(i) of Proposition 4.2 the $x_{0}$ variable is superfluous, but it is relevant in Part (a)(ii).
(ii) If we make the convention that $B_{0, n} \equiv 0$ a.s., put $u=0$ in (B.8), and identify $\left(J_{i}(0)\right)$ with a sequence $\left(L_{i}\right)$ of independent and identically distributed random
variables each having the distribution defined in (3.J)), we get the case $r=0$ of (4.7); namely, for $x_{k} \geq 1,0 \leq k \leq n-1, n=2,3, \ldots$, and $z>0$,

$$
\begin{equation*}
\mathrm{P}\left(\frac{\Delta_{t}^{(k)}}{\Delta_{t}^{(n)}}>x_{k}, 1 \leq k \leq n-1 \mid \Delta_{t}^{(n)}=\bar{\Pi}^{\leftarrow}(z / t)\right) \rightarrow \mathrm{P}\left(L_{n-1}^{(k)}>x_{k}, 1 \leq k \leq n-1\right), \tag{4.8}
\end{equation*}
$$

as $t \downarrow 0$, where $L_{n-1}^{(1)} \geq L_{n-1}^{(2)} \ldots \geq L_{n-1}^{(n-1)}$ are the decreasing order statistics of $\left(L_{i}\right)_{1 \leq i \leq n-1}$. Equation (LL.8) can of course be proved directly.
(iii) The case $r \in \mathbb{N}, n=1$, in Part(a) (i) of Proposition 4.2 , is covered by setting $n=r+1$, and $x_{1}=\cdots=x_{r-1}=1$ when $r>1$, in (4.8), to get

$$
\begin{equation*}
\lim _{t \downarrow 0} \mathrm{P}\left(\left.\frac{\Delta_{t}^{(r)}}{\Delta_{t}^{(r+1)}}>x_{r} \right\rvert\, \Delta_{t}^{(r+1)}=\bar{\Pi}^{\leftarrow}(z / t)\right)=\mathrm{P}\left(L_{r}^{(r)}>x_{r}\right)=x_{r}^{-r \alpha}, \tag{4.9}
\end{equation*}
$$

for $x_{r} \geq 1$ and $z>0$. Here $L_{r}^{(r)} \stackrel{\mathrm{D}}{=} \min _{1 \leq i \leq r} L_{i}$, where $\left(L_{i}\right)_{1 \leq i \leq r}$ are i.i.d. random variables, each having the distribution in (B.I⿴囗). Note that $L_{r}^{(r)} \stackrel{\mathrm{D}}{=} \mathrm{B}_{r, 1}^{-1 / \alpha}$.
(iv) Convergence of the conditional distributions in (4.7), (4.8), and (4.7), together with

$$
\begin{equation*}
\lim _{t \downarrow 0} \mathrm{P}\left(t \bar{\Pi}\left(\Delta_{t}^{(r+j)}\right) \leq z_{r+j}, 0 \leq j \leq n\right)=\mathrm{P}\left(\Gamma_{r+j} \leq z_{r+j}, 0 \leq j \leq n\right) \tag{4.10}
\end{equation*}
$$

for $0 \leq z_{r} \leq \cdots \leq z_{r+n}$, implies convergence of the corresponding joint, and hence marginal, distributions. Since the right-hand sides of (4.7), (4.8), and (4.9) do not depend on $z$, independence obtains in the corresponding limiting joint distributions. To verify ( 4.10 ), observe we have for $j=0,1, \ldots, n$,

$$
t \bar{\Pi}\left(\Delta_{t}^{(r+j)}\right) \stackrel{\mathrm{D}}{=} t \bar{\Pi}\left(\bar{\Pi}^{\leftarrow}\left(\Gamma_{r+j} / t\right)\right) \sim \frac{\bar{\Pi}\left(\bar{\Pi}^{\leftarrow}\left(\frac{\Gamma_{r+j}}{t}\right)\right)}{\bar{\Pi}\left(\bar{\Pi}^{\leftarrow}\left(\frac{1}{t}\right)\right)} \sim \frac{\Gamma_{r+j} / t}{1 / t}=\Gamma_{r+j},
$$

where the convergence is almost sure as $t \downarrow 0$.
Proposition 4.3 (Ratios smaller than 1). Suppose $\bar{\Pi}(\cdot) \in R V_{0}(-\alpha)$ with $0<\alpha<\infty$ and $r, n \in \mathbb{N}$.
(i) For each $z>0$ and $w \in(0,1)$

$$
\begin{equation*}
\lim _{t \downarrow 0} \mathrm{P}\left(W_{r, n}(t) \leq w \mid \Delta_{t}^{(r+n)}=\bar{\Pi}^{\leftarrow}(z / t)\right)=\mathrm{P}\left(K_{r+n-1}^{(n)} \leq w\right), \tag{4.11}
\end{equation*}
$$

where $K_{r+n-1}^{(n)}$ is the $n$th largest of $r+n-1$ independent and identically distributed random variables $\left(K_{i}\right)_{1 \leq i \leq r+n-1}$, with distribution $\mathrm{P}\left(K_{1} \leq w\right)=w^{\alpha}, w \in(0,1)$.
(ii) For each $z>0$ and $w \in(0,1)$

$$
\begin{equation*}
\lim _{t \downarrow 0} \mathrm{P}\left(\Delta_{t}^{(r)} \geq \bar{\Pi}^{\leftarrow}(z / t) \mid W_{r, n}(t)=w\right)=\mathrm{P}\left(\Gamma_{r+n} \leq w^{-\alpha} z\right) \tag{4.12}
\end{equation*}
$$

Remark 4.3. (i) Taking expectations in (4.لत) gives, as $t \downarrow 0$,

$$
W_{r, n}(t)=\frac{\Delta_{t}^{(r+n)}}{\Delta_{t}^{(r)}} \xrightarrow{\mathrm{D}} K_{r+n-1}^{(n)} .
$$

 we thus have the various alternatives

$$
W_{r, n}(t) \stackrel{\mathrm{D}}{\longrightarrow} W_{r, n} \stackrel{\mathrm{D}}{=} B_{r, n}^{1 / \alpha} \stackrel{\mathrm{D}}{=} K_{r+n-1}^{(n)} \stackrel{\mathrm{D}}{=} 1 / L_{r+n-1}^{(r)}, \text { as } t \downarrow 0,
$$

where $L_{r+n-1}^{(r)}$ is the $r$ th largest of i.i.d. $\operatorname{rvs}\left(L_{i}\right)_{1 \leq i \leq r+n-1}$.

Proof of Proposition 4.3: (i) The probability on the LHS of (4.

$$
\mathrm{P}\left(\Delta_{t}^{(r)} \geq w^{-1} \bar{\Pi}^{\leftarrow}(z / t) \mid \Delta_{t}^{(r+n)}=\bar{\Pi}^{\leftarrow}(z / t)\right)
$$

Conditional on $\left\{\Delta_{t}^{(r+n)}=\bar{\Pi}^{\leftarrow}(z / t)\right\}$, the ordered $\Delta_{t}^{(1)} \geq \cdots \geq \Delta_{t}^{(r+n-1)}$ have the distribution of the decreasing order statistics $K_{r+n-1}^{(1)}(t, z) \geq \cdots \geq K_{r+n-1}^{(r+n-1)}(t, z)$ of $r+n-1$ independent and identically distributed random variables $\left(K_{i}(t, z)\right)_{i=1,2, \ldots}$, each having the distribution

$$
\mathrm{P}\left(K_{1}(t, z) \in \mathrm{d} x\right)=\frac{\Pi(\mathrm{d} x) \mathbf{1}\left\{x \geq \bar{\Pi}^{\leftarrow}(z / t)\right\}}{\bar{\Pi}\left(\bar{\Pi}^{\leftarrow}(z / t)-\right)}, x>0
$$

From (2.4) it follows that, as $t \downarrow 0$, for each $w \in(0,1)$ and $z>0$,

$$
\begin{aligned}
\mathrm{P}\left(K_{1}(t, z)>w^{-1} \bar{\Pi}^{\leftarrow}(z / t)\right) & =\frac{t \bar{\Pi}\left(w^{-1} \bar{\Pi}^{\leftarrow}(z / t)-\right) \mathbf{1}\left\{w^{-1}>1\right\}}{t \bar{\Pi}\left(\bar{\Pi}^{\leftarrow}(z / t)-\right)} \\
& \rightarrow \frac{w^{\alpha} z \mathbf{1}_{\{w<1\}}}{z} \\
& =w^{\alpha} \mathbf{1}_{\{w<1\}}=\mathrm{P}\left(K_{1} \leq w\right),
\end{aligned}
$$

where $K_{1}$ is a random variable such that $K_{1}^{\alpha}$ is $U[0,1]$. Thus as $t \downarrow 0$, for $0<w<1$,

$$
\begin{aligned}
& \mathrm{P}\left(\Delta_{t}^{(r)} \geq w^{-1} \bar{\Pi}^{\leftarrow}(z / t) \mid \Delta_{t}^{(r+n)}=\bar{\Pi}^{\leftarrow}(z / t)\right) \\
& =\mathrm{P}\left(\text { at least } r \text { of } K_{1}(t, z), \ldots, K_{r+n-1}(t, z) \text { exceed or equal } w^{-1} \bar{\Pi}^{\leftarrow}(z / t)\right) \\
& =\sum_{k=r}^{r+n-1}\binom{r+n-1}{k}\left(\frac{\bar{\Pi}\left(w^{-1} \bar{\Pi}^{\leftarrow}(z / t)-\right)}{\bar{\Pi}\left(\bar{\Pi}^{\leftarrow}(z / t)-\right)}\right)^{k}\left(1-\frac{\bar{\Pi}\left(w^{-1} \bar{\Pi}^{\leftarrow}(z / t)-\right)}{\bar{\Pi}\left(\bar{\Pi}^{\leftarrow}(z / t)-\right)}\right)^{r+n-1-k} \\
& \rightarrow \sum_{k=r}^{r+n-1}\binom{r+n-1}{k}\left(w^{\alpha}\right)^{k}\left(1-w^{\alpha}\right)^{r+n-1-k} \\
& =\mathrm{P}\left(\text { at least } r \text { of } K_{1}, \ldots, K_{r+n-1} \text { are smaller than } w\right),
\end{aligned}
$$

and this is the RHS of (4.D⿴).
(ii) Using (2.2) suggests writing the LHS of (4.12) as

$$
\mathrm{P}\left(t \bar{\Pi}\left(\bar{\Pi}^{\leftarrow}\left(\Gamma_{r} / t\right)\right) \leq z \mid t \bar{\Pi}\left(\bar{\Pi}^{\leftarrow}\left(\Gamma_{r}+\widetilde{\Gamma}_{n}\right) / t\right)=t \bar{\Pi}\left(w \bar{\Pi}^{\leftarrow}\left(\Gamma_{r} / t\right)\right)\right),
$$

where $\widetilde{\Gamma}_{n}$ is a $\operatorname{Gamma}(n, 1)$ rv independent of $\Gamma_{r}$, a $\operatorname{Gamma}(r, 1)$ rv. By (L2.4) it is plausible that this tends to $\mathrm{P}\left(\Gamma_{r} \leq z \mid \Gamma_{r}+\widetilde{\Gamma}_{n}=w^{-\alpha} \Gamma_{r}\right)$ as $t \downarrow 0$. To prove it, write, to be brief, $A_{t}:=t \bar{\Pi}\left(\bar{\Pi}^{\leftarrow}\left(\Gamma_{r} / t\right)\right), W_{t}:=W_{r, n}(t)$ and $W:=W_{r, n}$, and let $C$ be any Borel subset of $\mathbb{R}^{+}$. Then, for each $t>0$, by dominated convergence,

$$
\begin{aligned}
& \int_{w \in C} \mathrm{P}\left(A_{t} \leq z \mid W_{t}=w\right) \mathrm{P}(W \in \mathrm{~d} w) \\
& =\int_{w \in C} \lim _{\varepsilon \downarrow 0}\left(\frac{\mathrm{P}\left(A_{t} \leq z, w-\varepsilon<W_{t}<w+\varepsilon\right)}{\mathrm{P}\left(w-\varepsilon<W_{t}<w+\varepsilon\right)}\right) \mathrm{P}(W \in \mathrm{~d} w) \\
& =\lim _{\varepsilon \downarrow 0} \int_{w \in C}\left(\frac{\mathrm{P}\left(A_{t} \leq z, w-\varepsilon<W_{t}<w\right)}{\mathrm{P}\left(w-\varepsilon<W_{t}<w\right)}\right) \mathrm{P}(W \in \mathrm{~d} w) .
\end{aligned}
$$

Since $\left(A_{t}, W_{t}\right)$ converges in distribution to $\left(\Gamma_{r}, W\right)$, as $t \downarrow 0$, and the limit distribution is continuous, the convergence is uniform. So, given $\delta>0$, there is a $t_{0}=t_{0}(\varepsilon, \delta)>0$ such that the last expression is, for $0<t \leq t_{0}$, no greater than

$$
\begin{aligned}
& \limsup _{\varepsilon \downarrow 0} \int_{w \in C}\left(\frac{\mathrm{P}\left(\Gamma_{r} \leq z, w-\varepsilon<W<w+\varepsilon\right)}{\mathrm{P}(w-\varepsilon<W<w+\varepsilon)}+\delta\right) \mathrm{P}(W \in \mathrm{~d} w) \\
& \leq \int_{w \in C} \mathrm{P}\left(\Gamma_{r} \leq z \mid W=w\right) \mathrm{P}(W \in \mathrm{~d} w)+\delta .
\end{aligned}
$$

In a similar way we find a lower bound for the liminf with $-\delta$, hence, for $0<t \leq t_{0}$,

$$
\left|\int_{w \in C}\left(\mathrm{P}\left(A_{t} \leq z \mid W_{t}=w\right)-\mathrm{P}\left(\Gamma_{r} \leq z \mid W=w\right)\right) \mathrm{P}(W \in \mathrm{~d} w)\right| \leq \delta .
$$

Now, choosing Borel sets $C^{+}$and $C^{-}$on which the integrand in the last integral is positive or negative, we see that

$$
\begin{equation*}
\int_{w>0}\left|\mathrm{P}\left(A_{t} \leq z \mid W_{t}=w\right)-\mathrm{P}\left(\Gamma_{r} \leq z \mid W=w\right)\right| \mathrm{P}(W \in \mathrm{~d} w) \leq \delta \tag{4.13}
\end{equation*}
$$

for $0<t \leq t_{0}$. Take any sequence $t_{k} \downarrow 0$. Let $k \rightarrow \infty$ and use Fatou's lemma, then let $\delta \downarrow 0$, to deduce from (4..3]) that

$$
\liminf _{k \rightarrow \infty} \mathrm{P}\left(A_{t_{k}} \leq z \mid W_{t_{k}}=w\right)=\mathrm{P}\left(\Gamma_{r} \leq z \mid W=w\right)
$$

(a.e. $w$ with respect to the distribution of $W$, thus, Lebesque a.e.). Taking a further subsequence if necessary, we can replace "liminf" by "lim" here. Then, since the limit holds for arbitrary $t_{k}$, we conclude that

$$
\lim _{t \downarrow 0} \mathrm{P}\left(A_{t} \leq z \mid W_{t}=w\right)=\mathrm{P}\left(\Gamma_{r} \leq z \mid W=w\right), \text { a.e. }(w)
$$

We can evaluate the probability on the RHS here using $W \stackrel{\mathrm{D}}{=}\left\{\left(\Gamma_{r}+\widetilde{\Gamma}_{n}\right) / \Gamma_{r}\right\}^{-1 / \alpha}$ and

$$
\mathrm{P}\left(\Gamma_{r} \leq z, \Gamma_{r}+\widetilde{\Gamma}_{n} \geq w^{-\alpha} \Gamma_{r}\right)=\int_{0 \leq v \leq z} \int_{x \geq w^{-\alpha} v-v}\left(\frac{e^{-x} x^{n-1}}{\Gamma(n)}\right) \mathrm{d} x\left(\frac{e^{-v} v^{r-1}}{\Gamma(r)}\right) \mathrm{d} v
$$

Differentiate with respect to $w$ and divide by the density of $W_{r, n}$ to get the required limiting conditional distribution as

$$
\frac{1}{f_{W_{r, n}}(w)} \int_{v=0}^{z}\left(\frac{e^{-\left(w^{-\alpha}-1\right) v}\left(w^{-\alpha}-1\right)^{n-1} \alpha w^{-\alpha-1} v^{n}}{\Gamma(n)}\right)\left(\frac{e^{-v} v^{r-1}}{\Gamma(r)}\right) \mathrm{d} v
$$

Substituting for $f_{W_{r, n}}(w)$ from (4.5), we can calculate the last expression as

$$
\int_{0}^{z} w^{-\alpha(r+n)} \frac{e^{-w^{-\alpha} v} v^{r+n-1}}{\Gamma(r+n)} \mathrm{d} v=\int_{0}^{w^{-\alpha} z} e^{-v} \frac{v^{r+n-1}}{\Gamma(r+n)} \mathrm{d} v
$$

which is the RHS of (4.12).
Proposition 4.4 (Ratios greater than 1). Let $\left\{\Gamma_{j}^{-1 / \alpha}, j \geq 1\right\}$ be the ordered points of a Poisson point process with intensity measure $\Lambda(\mathrm{d} x)=\alpha x^{-\alpha-1} \mathrm{~d} x \mathbf{1}_{x>0}, \alpha>0$. Let $r, n \in \mathbb{N}, 0<u<1, \lambda>0$. Then we have the conditional Laplace transform

$$
\begin{equation*}
\mathrm{E}\left(\left.\exp \left(-\lambda \sum_{i=1}^{n-1}\left(\frac{\Gamma_{r+i}}{\Gamma_{r+n}}\right)^{-1 / \alpha}\right) \right\rvert\,\left(\frac{\Gamma_{r+n}}{\Gamma_{r}}\right)^{-1 / \alpha}=u\right)=(\Phi(\lambda, u))^{n-1} \tag{4.14}
\end{equation*}
$$

where

$$
\Phi(\lambda, u)=\frac{\int_{1}^{1 / u} e^{-\lambda x} \Lambda(\mathrm{~d} x)}{1-u^{\alpha}}
$$

When $r=0$, the sum of ratios of jumps greater than 1 has representation

$$
\begin{equation*}
\sum_{i=1}^{n-1}\left(\frac{\Gamma_{i}}{\Gamma_{n}}\right)^{-1 / \alpha} \stackrel{\mathrm{D}}{=} \sum_{i=1}^{n-1} L_{i} \tag{4.15}
\end{equation*}
$$

where the $L_{i}$ are i.i.d random variables each with the same distribution as $\Gamma_{1} / \Gamma_{2}$; namely, $\mathrm{P}\left(L_{1} \in \mathrm{~d} x\right)=\Lambda(\mathrm{d} x) \mathbf{1}_{\{x>1\}}$.

Remark 4.4. The representation (4.5.5) of the sum of ratios of the ordered jumps of a stable subordinator as a random walk in $n$ provides the impetus for further work in large trimming results in the spirit of the investigations in Buchmann, Maller \& Resnick (2016).

Proof of Proposition 4.4: Equality (4.14) can be read from the order statistics property of the homogeneous Poisson process when $r>0$ and from (5.प) when $r=0$.

## 5 Converse Results

Theorem [5.] gives converses to the previous results.
Theorem 5.1 (Converse Results: Ratios Bigger than 1.). Suppose, for some $r \in \mathbb{N}$, $n \in \mathbb{N}, \Delta_{t}^{(r)} / \Delta_{t}^{(r+n)} \xrightarrow{\mathrm{D}} Y$, as $t \downarrow 0$, for an extended value random variable ${ }^{\mathbb{D}} Y \geq 1$. Then one of the following holds:
(i) $\mathrm{P}(1<Y<\infty)>0$, in which case $\bar{\Pi}(\cdot) \in R V_{0}(-\alpha)$ with $0<\alpha<\infty$;
(ii) $\mathrm{P}(Y=1)=1$, in which case $\bar{\Pi}$ is rapidly varying at 0 ;
(iii) $Y=\infty$ a.s., in which case $\bar{\Pi}$ is slowly varying at 0 .

Remark 5.1. [Converse Results: Ratios Smaller than 1.] Analogous results to Theorem [.] for ratios smaller than 1 follow by taking reciprocals. Write $\Delta_{t}^{(r+n)} / \Delta_{t}^{(r)}=$ $\left(\Delta_{t}^{(r)} / \Delta_{t}^{(r+n)}\right)^{-1}$ and apply the theorem, replacing $Y$ by $Y^{-1}$, and making the obvious interpretations in Parts (i), (ii) and (iii) of the theorem.

Proof of Theorem 5.7: Assume for some $r \in \mathbb{N}, n \in \mathbb{N}$,

$$
\begin{equation*}
\frac{\Delta_{t}^{(r)}}{\Delta_{t}^{(r+n)}} \xrightarrow{\mathrm{D}} Y \text {, as } t \downarrow 0, \tag{5.1}
\end{equation*}
$$

where $Y$ is an extended random variable with distribution $G$, say, on $[1, \infty]$. The proof that follows is similar in style to that of Kevei \& Mason (2014) who considered ratios of successive jumps, that is, the case $n=1$. When $n>1$, some rather different arguments are needed at some places.

Keep $u$ fixed in $(0,1)$ throughout the remainder of the proof and use ( $[2.2)$ to write

$$
\begin{aligned}
\mathrm{P}\left(\Delta_{t}^{(r+n)}<u \Delta_{t}^{(r)}\right) & =\mathrm{P}\left(\bar{\Pi}^{\leftarrow}\left(\left(\Gamma_{r}+\widetilde{\Gamma}_{n}\right) / t\right)<u \bar{\Pi}^{\leftarrow}\left(\Gamma_{r} / t\right)\right) \\
& =\int_{y \geq 0} \mathrm{P}\left(\widetilde{\Gamma}_{n}>t \bar{\Pi}\left(u \bar{\Pi}^{\leftarrow}(y / t)\right)-y\right) \mathrm{P}\left(\Gamma_{r} \in \mathrm{~d} y\right),
\end{aligned}
$$

where $\Gamma_{r}$ and $\widetilde{\Gamma}_{n}$ are independent Gamma random variables. Substituting for their densities gives

$$
\begin{aligned}
& \mathrm{P}\left(\Delta_{t}^{(r+n)}<u \Delta_{t}^{(r)}\right)=\int_{y>0} \int_{z>t \bar{\Pi}\left(u \bar{\Pi}^{\leftarrow}{ }_{(y / t))-y}\right.}\left(\frac{e^{-z} z^{n-1}}{\Gamma(n)}\right) \mathrm{d} z\left(\frac{e^{-y} y^{r-1}}{\Gamma(r)}\right) \mathrm{d} y \\
& =\frac{t^{r+n}}{\Gamma(r+n)} \int_{y>0} \int_{z>\bar{\Pi}\left(u \bar{\Pi}^{\leftarrow}(y)\right)}\left(\frac{e^{-t z}(z-y)^{n-1}}{B(r, n)}\right) \mathrm{d} z y^{r-1} \mathrm{~d} y
\end{aligned}
$$

[^1]\[

$$
\begin{equation*}
=\frac{t^{r+n}}{\Gamma(r+n)} \int_{z>0} \int_{y<\bar{\Pi}\left(\bar{\Pi}^{\leftarrow}(z) / u\right) / z}\left(\frac{(1-y)^{n-1} y^{r-1}}{B(r, n)}\right) \mathrm{d} y e^{-t z} z^{r+n-1} \mathrm{~d} z . \tag{5.2}
\end{equation*}
$$

\]

Here note that, since $u<1$, we have $t \bar{\Pi}\left(u \bar{\Pi}^{\leftarrow}(y / t)\right) \geq t \bar{\Pi}\left(\bar{\Pi}^{\leftarrow}(y / t)-\right) \geq y$, and $\bar{\Pi}\left(\bar{\Pi}^{\leftarrow}(z) / u\right) / z \leq \bar{\Pi}\left(\bar{\Pi}^{\leftarrow}(z)\right) / z \leq 1$. We recognise the inner integral in ( $\left.\overline{5}, 2\right)$ as the incomplete Beta function $B\left(r, n ; \bar{\Pi}\left(\bar{\Pi}^{\leftarrow}(z) / u\right) / z\right)$ (see ( 2 [23)) ). By assumption (5.ل), the expression in (5.2) tends to $\bar{G}(1 / u):=P(Y>1 / u)$ as $t \downarrow 0$, at continuity points of $\bar{G}$. To simplify the notation, from this point on let $x=1 / u>1$. Let

$$
U_{x}(z):=\int_{0}^{z} B\left(r, n ; \bar{\Pi}\left(\bar{\Pi}^{\leftarrow}(v) x\right) / v\right) v^{r+n-1} \mathrm{~d} v, z>0 .
$$

Then from (5.2),

$$
t^{r+n} \int_{z>0} e^{-t z} U_{x}(\mathrm{~d} z) \rightarrow \Gamma(r+n) \bar{G}(x), \text { as } t \downarrow 0
$$

at continuity points of $\bar{G}$, which by Thm. 1.7 .1 p .37 of Bingham et al. (1987) implies

$$
z^{-r-n} U_{x}(z) \rightarrow \Gamma(r+n) \bar{G}(x) / \Gamma(r+n+1)=\bar{G}(x) /(r+n), \text { as } z \rightarrow \infty .
$$

Write this as

$$
\begin{equation*}
\frac{1}{z^{r+n}} \int_{0}^{z} b_{x}(v) v^{r+n-1} \mathrm{~d} v \rightarrow \frac{\bar{G}(x)}{r+n}, \text { as } z \rightarrow \infty, \tag{5.3}
\end{equation*}
$$

where

$$
b_{x}(v)=B\left(r, n ; f_{x}(v)\right)=\frac{1}{B(r, n)} \int_{0}^{f_{x}(v)} y^{r-1}(1-y)^{n-1} \mathrm{~d} y=: \int_{0}^{f_{x}(v)} p(y) \mathrm{d} y
$$

with $f_{x}(v):=\bar{\Pi}\left(\bar{\Pi}^{\leftarrow}(v) x\right) / v, v>0$ and $p(y):=y^{r-1}(1-y)^{n-1} / B(r, n), 0 \leq y \leq 1$. Note that $x$ is kept fixed in $b_{x}(v)$ and $f_{x}(v)$. We have $0 \leq f_{x}(v) \leq 1$, so $0 \leq b_{x}(v) \leq 1$, for all $v>0$. (5.3) implies

$$
\begin{equation*}
\frac{1}{z^{r+n}} \int_{z}^{\lambda z} b_{x}(v) v^{r+n-1} \mathrm{~d} v=\int_{1}^{\lambda} b_{x}(v z) v^{r+n-1} \mathrm{~d} v \rightarrow \frac{\left(\lambda^{r+n}-1\right) \bar{G}(x)}{r+n}, \text { as } z \rightarrow \infty, \tag{5.4}
\end{equation*}
$$

for any $\lambda>1$ and each fixed $x>0$.
Functions $b_{x}(v), f_{x}(v)$, are not necessarily monotone but are of bounded variation (BV) on finite intervals bounded away from 0 . To see this, observe that the function $m_{x}(v):=v f_{x}(v)=\bar{\Pi}\left(\bar{\Pi}^{\leftarrow}(v) x\right)$ is nondecreasing in $v$ and

$$
\left|\mathrm{d} f_{x}(v)\right|=\left|\frac{\mathrm{d} m_{x}(v)}{v}-\frac{m_{x}(v) \mathrm{d} v}{v^{2}}\right| \leq \frac{\mathrm{d} m_{x}(v)}{v}+\frac{\mathrm{d} v}{v} ;
$$

thus, with $p_{0}:=\sup _{0 \leq y \leq 1} p(y)$,

$$
\left|\mathrm{d} b_{x}(v)\right|=\left|p\left(f_{x}(v)\right) \mathrm{d} f_{x}(v)\right| \leq p_{0}\left(\frac{\mathrm{~d} m_{x}(v)}{v}+\frac{\mathrm{d} v}{v}\right),
$$

and the RHS is integrable over $v \in[\delta, z]$, for any $0<\delta<z$. So $f_{x}$ and $b_{x}$ are of bounded variation on $[\delta, z]$ for any $0<\delta<z$. Take any sequence $z_{k} \rightarrow \infty$. By Helly's theorem for finite measures we can find a subsequence, also denoted $z_{k}$, possibly depending on $x$, such that

$$
b_{x}\left(v z_{k}\right) \rightarrow g_{x}(v), v>0, \text { as } k \rightarrow \infty,
$$

at continuity points of $g$, for a function $g_{x}(v) \in[0,1]$. Using dominated convergence in (5.4) we get

$$
\int_{1}^{\lambda} g_{x}(v) v^{r+n-1} \mathrm{~d} v=\frac{\left(\lambda^{r+n}-1\right) \bar{G}(x)}{r+n}=\bar{G}(x) \int_{1}^{\lambda} v^{r+n-1} \mathrm{~d} v .
$$

This holds for all $\lambda>1$ and so implies $g_{x}(v)=\bar{G}(x)$, for all $v>1, x>0$, not depending on the choice of subsequence. Thus we deduce that

$$
b_{x}(v z)=\int_{0}^{f_{x}(v z)} p(y) \mathrm{d} y \rightarrow \bar{G}(x)
$$

as $z \rightarrow \infty$, at continuity points of $\bar{G}$, for all $v>1$. Take $v=2$. Now $f_{x}(2 z)$ is monotone in $x$ for each $z$, so by Helly's theorem again each sequence $z_{k} \rightarrow \infty$ contains a further subsequence, also denoted $z_{k}$, such that $f_{x}\left(2 z_{k}\right) \rightarrow h(x) \in[0,1]$, as $k \rightarrow \infty$, at continuity points of $h(x)$. Thus we obtain

$$
\begin{equation*}
\int_{0}^{h(x)} p(y) \mathrm{d} y=\bar{G}(x) \tag{5.5}
\end{equation*}
$$

at continuity points of $h$. Again the limit does not depend on the choice of subsequence. This identifies $h(x)$ as $I^{\leftarrow}(\bar{G}(x))$, where $I^{\leftarrow}(\cdot)$ is the unique inverse function to the continuous strictly increasing function $I(\cdot)=\int_{0}^{\cdot} p(y) \mathrm{d} y$. Thus, continuity points of $h$ are points of increase of $G$. Define

$$
\mathcal{A}:=\{x \geq 1: x \text { is a continuity point and a point of increase of } G\} .
$$

We conclude that

$$
\begin{equation*}
\lim _{z \rightarrow \infty} \frac{\bar{\Pi}\left(\bar{\Pi}^{\leftarrow}(z) x\right)}{z}=\lim _{z \rightarrow \infty} f_{x}(z)=\lim _{z \rightarrow \infty} f_{x}(2 z)=h(x), \text { for all } x \in \mathcal{A}, \tag{5.6}
\end{equation*}
$$

where $h$ satisfies (5.5). (5.6) is exactly analogous to Eq. (2.10) of Kevei \& Mason (2014) and we follow their arguments henceforth to finish the converse part of the proof. There are three alternatives.
(i) $\mathrm{P}(1<Y<\infty)>0$. In this case $\bar{G}$ has at least one point of decrease in $(1, \infty)$, say $x$, and a neighbourhood $(x-\varepsilon, x+\varepsilon)$ for some $\varepsilon>0$, such that $\bar{G}(y)>0$ for all $y$ in the neighbourhood. Kevei \& Mason (2014) gave a careful analysis of this situation, showing that it leads to $\bar{\Pi}(\cdot) \in R V_{0}(-\alpha)$ with $0<\alpha<\infty$.
(ii) $\mathrm{P}(Y=1)=1$. This means that $\mathrm{P}(Y>x)=\bar{G}(x)=0$ for all $x>1$, so $\int_{0}^{h(x)} p(y) \mathrm{d} y=0$ and

$$
\lim _{z \rightarrow \infty} \frac{\bar{\Pi}\left(\bar{\Pi}^{\leftarrow}(z) x\right)}{z} \leq \lim _{z \rightarrow \infty} \frac{\bar{\Pi}\left(\bar{\Pi}^{\leftarrow}(z) x\right)}{\bar{\Pi}\left(\bar{\Pi}^{\leftarrow}(z)\right)}=0
$$

for all $x>1$. Thus $\bar{\Pi}$ is rapidly varying at 0 .
(iii) $Y=\infty$ a.s. This means that $\mathrm{P}(Y>x)=\bar{G}(x)=1$ for all $x>1$, so $\int_{0}^{h(x)} p(y) \mathrm{d} y=1$ and

$$
\lim _{z \rightarrow \infty} \frac{\bar{\Pi}\left(\bar{\Pi}^{\leftarrow}(z) x\right)}{z}=1
$$

for all $x>1$. This leads to $\bar{\Pi}$ slowly varying at 0 as shown in Kevei \& Mason (2014), and completes the proof.

## 6 Related Results: Order Statistics of i.i.d. rvs

We conclude with some history relating how these kinds of results have antecedents in the literature of order statistics of i.i.d. real-valued random variables. The general scenario there is of the order statistics $X_{n}^{(n)} \leq \cdots \leq X_{n}^{(1)}$ of i.i.d. rvs $\left(X_{i}\right)_{i \in \mathbb{N}}$ in $\mathbb{R}$ with distribution $F$ such that $F(x)<1$ for all $x$. The asymptotic is then as $n \rightarrow \infty$ ("large time"). (In most of the results quoted below the distribution $F$ is also assumed continuous, so ties among order statistics have probability 0 . We avoided such an assumption on $\Pi$ in our results.)

An early and well-cited venture in this area was by Arov \& Bobrov (1960). They considered not only the order statistics but also their sum, i.e., the random walk $S_{n}$ whose step sizes are the $X_{i}$, obtaining among other things results for convergence of joint distributions of deterministically normed order statistics, and as a corollary limiting distributions for ratios of (not necessarily successive) order statistics. This was extended to ratios of the sum after removal of a fixed number of extreme terms (i.e., the trimmed sum) to large order statistics. (The distribution $F$ was assumed to have a density.)

Smid \& Stam ([1975) considered the $\left(X_{n}^{(i)}\right)_{1 \leq i \leq n}$ as above, and, in what amounts to a generalisation of and converse to one of the Arov \& Bobrov (T96T) results, showed that $\lim _{n \rightarrow \infty} \mathrm{P}\left(X_{n}^{(j+1)} / X_{n}^{(j)} \leq x, 1 \leq j \leq k\right)=\prod_{j=1}^{k} x^{j \alpha}$ for all $x \in(0,1)$ and $k \in \mathbb{N}$ iff the distribution tail $\bar{F}(x) \in R V_{\infty}(-\alpha), \alpha \geq 0$. They include the $\alpha=0$ case ( $\bar{F}$ slowly varying at $\infty$ ). Their proof used Scheffé's lemma and applications of the Wiener-Tauberian theory. A converse to another of the Arov \& Bobrov (1960) results is in Maller \& Resnick ([9884). An earlier result along the lines of Smid \& Stam (1975) is in Shorrock (1972).

Teugels (198]I) considered order statistics of i.i.d. rvs in the domain of attraction of a stable law, and gave results extending some of the Arov-Bobrov limit laws concerning ratios of sums of order statistics to their (trimmed) sums. For an application of these kinds of ideas in reinsurance, see Ladoucette \& Teugels (2006).

Lanzinger \& Stadtmüller (2002) gave a simplified version of the Smid \& Stam (1.975) result (for the $k=1$ case) and extended this for when $F$ is in the domain of attraction of an extreme value distribution.

There is of course in addition a very large literature analysing various functions of order statistics of i.i.d. real-valued rvs which we do not attempt to summarise here.

We remark finally that while there are obvious correspondences between the (large-time) i.i.d. case and the (small time) point process case, there are significant differences too. One aspect is that, in view of our assumption $\bar{\Pi}(0+)=\infty$, there are always infinitely many points of the process in any right neighbourhood of 0 , hence, infinitely many ordered points; whereas, in the i.i.d. case, there are of course at most $n$ order statistics in a sample of size $n$. Thus there is no immediate counterpart of results like (3.61) or (4.3). This feature actually simplifies some of the point process proofs, for example that of Theorem [3.D, although the formulation is more complex.

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[^1]:    ${ }^{1}$ A random variable that may take the value $+\infty$ with positive probability.

