Multivariate Heavy Tails in Complex Networks

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Abstract. It has been shown that many real networks are not only "scale-free" (i.e., having a power-law degree distribution), but also contain more complex structures such as "hierarchy" or "self-similarity" that cannot be captured by the preferential attachment random network model. These observations have led to a number of more sophisticated models being proposed in the literature. In this paper we advocate a *multivariate analysis* perspective based on the notion of MRVs as a unifying framework to study complex structures in networks. We demonstrate the existence of "multivariate heavy tails" in existing network models and real networks, and argue that they better capture the "hierarchical" or "self-similar" structures in these networks.

1 INTRODUCTION

Complex networks arising from natural, social, and engineered systems have been a topic of extensive studies in the past decades. A prevailing feature characterizing most of these complex networks is the *power-law* degree distribution, $P(k) \approx k^{-\gamma}$, where k represents the node degree. This gives rise to the term *scale-free* (SF) networks. Generative network models such as *preferential attachment* (PA) model have been proposed to provide a plausible explanation for the origin of the power-law degree distribution (also the "small-world" phenomenon observed in such networks, characterized by the average shortest path length in the logarithmic (log) order of the network size, $O(\log N)$). Besides the power-law degree distribution and small-world properties, many real-world complex networks also exhibit other important features, such as modularity (e.g., as characterized by a high average clustering coefficient) [1] that is absent in random scale-free networks generated via the PA model. As an attempt to capture the more complex structures observed in real-world networks, many additional models (mostly *deterministic*) have been introduced in the literature, some of which yield sometimes confusing, if not contradictory, statements about the structures of complex networks. These earlier studies all share a *common characteristic* in their approaches to capture more complex structures in networks: in addition to the power-law degree distribution, they introduce – and look for – the *scaling law* in another metric or form, e.g., the clustering coefficients [1].

In this paper we bring a *multivariate analysis* perspective – in particular, the notion of *multivariate heavy tails* – to study structural properties of complex networks. As a generalization of the power law distributions in one dimension, "multivariate heavy tailed" (more precisely, *multivariate regularly varying* or *MRV* in short, see Section 3 for details) distributions embody more complex structures[1], and have been applied to a number of fields, e.g., multivariate time series analysis in finance to identify shared risks [7]. Like in the previous studies, this *multivariate analysis* perspective allows us to study the structural properties of complex networks using multiple metrics (that go beyond the degree distribution); but unlike the previous studies, it enables us to *explicitly and directly* examine the *dependence* structures defined by a number of different metrics, e.g., node degree and clustering coefficient, or degree-degree dependence structures, in complex networks. Such dependence structures cannot be revealed by studying each of the network statistical features in the marginal form alone. For example, intuitively "hierarchical" or "self-similar" structures introduce dependencies among sub-network structures at multiple scales and these dependences are all explicitly built through *recursive construction* in the growing network models introduced in [1, 5, 6, 3]. However, finding a scaling law in the marginal distribution in each of the metrics of interest (e.g., degrees and clustering coefficients) in a *real* network that matches those in the *synthetic* growing network models does not necessarily imply that the real network has the same dependence structures – at least theoretically speaking, those marginal heavy tails can occur *independently* in the network. Nonetheless, existence of MRVs in the joint distribution of these metrics provides a much stronger evidence for the "heavy-tailed" dependence structures. Hence we believe that existence of MRVs provides a better measure to capture the "hierarchical" or "self-similarity" structures in complex networks. Due to space limitation, we only provide a few examples in Section 4. In particular, we illustrate that the joint degree-clustering coefficient distribution in the synthetic growing networks of [1, 5] is " multivariate heavy-tailed." Furthermore, we show that these two models have *distinct* structures in that the model in [5] contains a MRV joint *degree-degree* distribution, whereas that in [1] does not.

In summary, in this paper we advocate a *multivariate analysis* perspective based on the notion of MRVs as a unifying framework to study complex structures in networks. To the best of our knowledge, we believe ours is the first to apply such a framework in the study of complex networks, and demonstrate the existence of "multivariate heavy tails" in synthetic and real networks.

2 BEYOND THE POWER-LAW DEGREE DISTRIBUTION: An Overview of Existing Models

It is known that the structures of many real complex networks are not completely random: they are highly modular, and some have some "self-repeating" hierarchical patterns; these complex structures cannot be captured by the scale-free random network models such as the preferential attachment (PA) model, as shown in [1, 4, 2, 3]. Several studies have attempted to capture this "self-similarity" or "hierarchy" either in the form of proposing deterministic graph models or in the form of suggesting a measure. They have common characteristic in their approaches by finding scaling law relations in different forms and different metrics, in addition to power-law degree distribution. In the following, we present a brief overview of the existing models.

[1] For example, a multivariate distribution can have a power-law *marginal* distribution in each variable, but not jointly MRV, i.e., multivariate heavy-tailed.

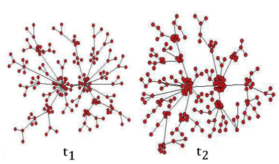


Fig. 1: Prefrential attachement model

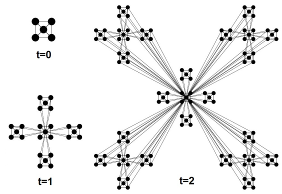


Fig. 2: Ravasaz *et al.*model

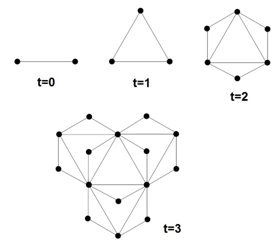


Fig. 3: Dorogovtsev *et al.*model

Ravasz and Barabasi [1] suggest the scaling law between the degree and clustering coefficient of the nodes $C(k) \sim k^{-\beta}$ as a quantity revealing the intrinsic hierarchy of the networks. They propose a deterministic graph model having this scaling law with an iterative construction leading to a hierarchical structure. The construction starts from a 5-vertex clique with one of the five nodes indicated as the center. It continues with replicating four copies of this cluster and connecting the center node of the central cluster to the peripheral nodes of the non-central clusters (see Fig.2). Combining the scale-free topology with high modularity, the hierarchical modularity of the model results in the aforementioned scaling law between the node degrees and clustering coefficients. The authors believe that the hierarchy in real networks is the result of combining many small but densely connected clusters to form larger but less cohesive groups, with this process repeating recursively. Dorogovtsev *et al.* [5] propose a different deterministic network model which obeys the same scaling law between the node degree and clustering coefficient, in addition to the power-law degree distribution. The recursive construction of this graph starts from an edge connecting two nodes and it grows by adding a node for every edge in the network and attaching it to both ends of the edge (see Fig.3). The random variation of this construction is creating a node per unit time and connecting it to both ends of a randomly chosen edge. Since this graph has no fixed finite fractal dimension, they call it a *pseudo-fractal* web. This network model also exhibits a strong short-range degree-degree correlation identified by $P(k,k') \sim k^{1-\gamma}k'^{-2}$. A number of other deterministic graph models have been proposed in the literature to capture the "self-similarity property" of real networks; all resort to a recursive procedure but with different constructions. Comellas *et al.* [8] generalizes the model in [5] by starting from a clique (q-vertex clique for any q, where $q = 2$ corresponds to [5]) and continuing by adding one node per clique in the network and attaching it to all nodes in the clique. Another example is by Zhang *et al.* [9] which uses the basic Sierpinski Gasket structure or a generalized form obtained by dividing the edges of the triangles to more than two pieces. They translate these fractal geometrical structures to graphs by assigning the nodes of the graph to downward pointing triangles, and making two nodes connected if the boundaries of the corresponding triangles have touching points, considering the three sides of the outer triangle as three different nodes as well. The networks constructed thereof all have power-law degree distributions and obey the scaling law between the node degrees and clustering coefficients.

In contrast to the above deterministic network models which obey the scaling law between the node degree and clustering coefficients, Chen *et al.* [6] present a deterministic network model with a power-law degree distribution, but all nodes have zero clustering coefficient. It is recursively constructed from square-shaped elements, thus all nodes having zero clustering coefficient. They state that their network models are consistent with some real networks such as electronic circuits and the Internet at the router level which have reduced clustering coefficients. Because of its "modular" and "hierarchical" construction, the authors argue that their network models provide a counter-proof that hierarchical organization of modularity in complex networks must obey a scaling law between the node degrees and clustering coefficients as claimed in [1, 4]. In addition to the studies discussed above, there are also a number of other studies either attempting to directly capture, e.g., "self-similar" structures in complex networks[2], or proposing other metrics, e.g., S-metric [11] , to capture degee-degree correlations in these networks. Due to space limitation, we do not elaborate them here.

3 MULTIVARIATE HEAVY TAILS: A Quick Primer

The theory of regularly varying functions is an essential analytical tool for dealing with heavy tails, long-range dependence and domains of attraction, see [7, 12]. Roughly speaking, regularly varying functions are those functions which behave asymptotically like "power-law" functions. In the following, we provide a quick introduction to *multivariate regularly varying* (MRV) functions; the interested reader is referred to [7] for more details. We conclude with a proposition and a method of our own which provide a sufficient condition and a convenient tool to check for the existence of MRVs empirically.

Definition 1. [7] A measurable function $U : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is *regularly varying* at ∞ with tail index $\alpha \in \mathbb{R}$ if for any $x > 0$,

$$\lim_{t\to\infty} \frac{U(tx)}{U(t)} = x^{\alpha}. \quad (1)$$

To check whether the distribution $F(x)$ of a single random variable X is regularly varying, the complementary cumulative distribution function (CCDF) is used in the definition above by substituting $U(x) = 1 - F(x)$ (for large values of x, $U(x)$ gives us the tail distribution). To get a better sense of eq. (1) it is easy to check that it holds for a random variable x having the Pareto distribution, as a special case of regularly varying distributions, with parameters c and α, and cumulative distribution function $F(x) = 1 - (\frac{c}{x})^{\alpha}$. The equivalent form of the definition above for a regularly varying distribution with a tail index of α is as follow [12]:

$$\lim_{t\to\infty} t\mathrm{P}(X > t^{\frac{1}{\alpha}} x) = x^{-\alpha}, \text{ for } x > 0. \quad (2)$$

Empirically in data analysis the most convenient (and visual) method to verify whether a random variable has a regularly varying distribution (i.e., it is "heavy-tailed") is to plot its CCDF in a log-log scale (*cf.* the *QQ-plot*) and check its linearity. This is what we will use in this paper also. For an instance, our analysis in fig. (5(a)) shows that the degree of nodes as random variable X, has regularly varying distribution because of its linear behavior in log-log scale of CCDF plot. But random variable y in fig. (9(b)) is not regularly varying.

Generalizing **Definition 1** to more than one random variable (or equivalently, to higher-dimensional measurable functions) gives us the notion of *multivariate regular variations* (MRVs). (We note that one cannot generalize the definition of "power-law" in eq.(2) to more than one-dimension in a straightforward manner.) A *necessary* condition for a multivariate distribution (measurable function) to be *multivariate regularly varying* is that all of its *marginal* distributions must be regularly varying. In the following we provide the definition of MRV measurable functions/distributions.

Definition 2. Consider a random vector $\mathbf{X}$ with dimension $d(\geq 1)$ and a cone $C \subset \mathbb{R}^d$, where $\mathbf{1} = (1, \ldots, 1) \in C$. We say a (nondecreasing) measurable function $U(\mathbf{x})$ defined on C, $U : C \mapsto [0, \infty)$, is *multivariate regularly varying* (MRV) on C with limit function $\lambda(\cdot)$ if $\lambda(\cdot) > 0$ and for all $\mathbf{x} \in C$ we have [7]

$$\lim_{t \to \infty} \frac{U(t\mathbf{x})}{U(t\mathbf{1})} = \lambda(\mathbf{x}). \tag{3}$$

Since in practical data analysis we often deal with *bivariate* distributions, we provide the equivalent form of the conditions for a random vector of two variables (X, Y) to be MRV as below:
i) X has a regularly varying distribution with tail index of α,
ii) the distribution of Y is regularly varying with tail index β,
iii) the limit function $\mu(\cdot) > 0$ does exist for any choice of $(x, y) \in (\mathbf{0}, \infty)$, in the following relation

$$\lim_{t \to \infty} t\mathrm{P}(X > t^{1/\alpha} x, Y > t^{1/\beta} y) = \mu(x, y). \tag{4}$$

In empirical data analysis conditions (i) and (ii) can be checked using the standard log-log plot of the CCDF (or the Q-Q plot) of each variable. However, there is in general no easy visual tool to check for condition (iii). In the following we show that it suffices to verify condition (iii) by checking for the existence of the limit function along a particular line $(\check{x}, r\check{x})$ for a fixed $\check{x}$ and $r \in (0, \infty)$ in the cone $(\mathbf{0}, \infty)$.

Proposition. Assume that conditions (i) and (ii) hold true and that $\mu(\check{x}, r\check{x})$ in eq.(4) exists for a fixed $\check{x} > 0$ and all $r \in (0, \infty)$. Then the limit $\mu(x, y)$ in eq.(4) exists for all possible pairs of $(x, y) \in (\mathbf{0}, \infty)$.

Proof. First we note that the existence of the limit $\mu(x, y)$ along the X- or Y- axes $\{(x, 0), x > 0\}$, or $\{(0, y), y > 0\}$ is guaranteed by condition (i) or (ii). Hence we only need to consider the cases $x > 0, y > 0$. For a fixed $\check{x} > 0$, we can rewrite (x, y) in the form of $(c_1\check{x}, c_2\check{x})$, where $c_1 := x/\check{x} > 0$ and $c_2 := y/\check{x} > 0$. Then we have

$$\begin{aligned}
\lim_{t\to\infty} t\mathrm{P}(X > t^{\frac{1}{\alpha}}x, Y > t^{\frac{1}{\beta}}y) &= \lim_{t\to\infty} t\mathrm{P}(X > t^{\frac{1}{\alpha}}c_1\check{x}, Y > t^{\frac{1}{\beta}}c_2\check{x}) \\
&= \lim_{t\to\infty} t\mathrm{P}(X > (c_1^{\alpha}t)^{\frac{1}{\alpha}}\check{x}, Y > (c_1^{\alpha}t)^{\frac{1}{\beta}}\frac{c_2}{c_1^{\alpha/\beta}}\check{x}) \\
&= \frac{1}{c_1^{\alpha}}\lim_{t'\to\infty} t'\mathrm{P}(X > t'^{\frac{1}{\alpha}}\check{x}, Y > t'^{\frac{1}{\beta}}\frac{c_2}{c_1^{\alpha/\beta}}\check{x}) \\
&= \frac{1}{c_1^{\alpha}}\mu(\check{x}, r\check{x}) > 0, \quad \text{where } r = c_2/c_1^{\alpha/\beta} > 0.
\end{aligned} \tag{5}$$

Without loss of generality, we set $\check{x} = 1$ throughout our analyses. Hence, this proposition reduces condition (iii) that the limit in eq. (4) exists for the entire *quadrant* $\{(x,y) \in (\mathbf{0},\infty)\}$ to that along the *half-line* $\{(1,r) : r \in (0,\infty)\}$. However, checking the existence of the limit (of a bivariate function) along this half-line is still not straightforward, especially when applying to practical data practical. In the following we transform this problem into that of checking a *parametrized* family of *univariate* random variables $Z(r)$ are regularly varying. Substituting $(x,y) = (1,r)$ into eq.(4), we have

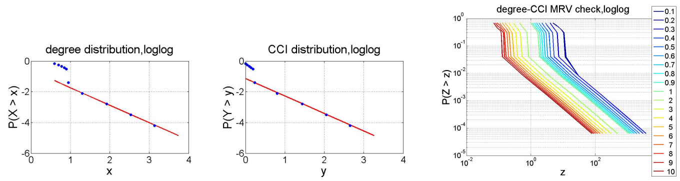
$$\begin{aligned}
\lim_{t\to\infty} t\mathrm{P}(X > t^{\frac{1}{\alpha}}, Y > t^{\frac{1}{\beta}}r) &= \lim_{t\to\infty} t\mathrm{P}(X^{\alpha}r^{\beta} > tr^{\beta}, Y^{\beta} > tr^{\beta}) \\
&= \lim_{t\to\infty} t\mathrm{P}(\min\{X^{\alpha}r^{\beta}, Y^{\beta}\} > tr^{\beta}) \\
&= \lim_{t\to\infty} t\mathrm{P}(Z(r) > tz),
\end{aligned} \tag{6}$$

where the (univariate) random variable $Z(r) := \min\{X^{\alpha}r^{\beta}, Y^{\beta}\}$ and $z = r^{\beta}$. Hence if $Z(r)$ is regular varying with the tail index $\gamma = 1$ (*cf.*, eq.(2)), i.e., $\lim_{t\to\infty} t\mathrm{P}(Z(r) > tz) = z^{-1}$ for any $z > 0$, then eq.(6) holds for $z = r^{\beta}$.

The above results yield a convenient *direct* and *visual* tool to check for the existence of *bivariate heavy tails* when performing empirical data analysis. Using the same log-log plotting procedure described earlier for empirically checking the existence of univariate RVs, we first compute $Z(r)$ from the data for a range of different values of r, then plot the CCDF of $Z(r)$ in the log-log scale – in a sense this produces a form of (log-log) "contour" plot of the random variables $Z(r)$ – and check for the linearity of the "contours." For large $z > 0$, more linear all the contours appear, stronger the emprical evidence suggests the existence of MRVs in the data. In the next section we provide several examples as illustrations.

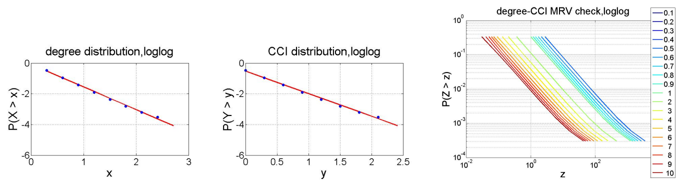
4 MRV IN NETWORKS

We now apply the theory of MRVs to the existing "hierarchical" and other network models proposed in the literature. Inspite of their different constructions, a common characteristic of these models share is that they contain *multivariate heavy tails* in one form or another. We also demonstrate that a number of real networks also contain complex structures indicative of multivariate heavy tails. This suggests the theory of MRVs as a *unifying* framework to study the more complex structures in networks.



(a) Mariginal deg distr. (b) Mariginal CCI distr. (c) $Z(r)$ plots for joint deg-CCI distr.

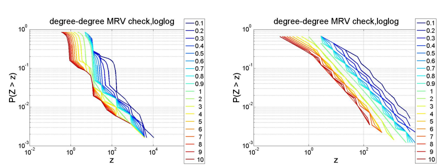
Fig. 4: MRV Analysis for the Ravasz *et al.* model.



(a) Mariginal deg distr. (b) Mariginal CCI distr. (c) $Z(r)$ plots for joint deg-CCI distr.

Fig. 5: MRV Analysis for the Dorogovtsev *et al.* model.

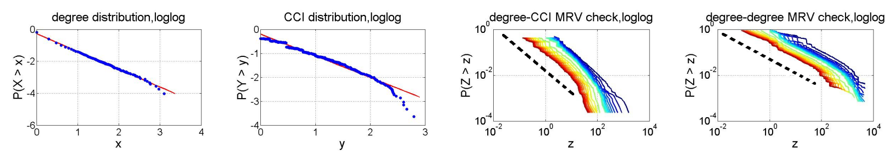
We first consider the deterministic "hierarchical" network model proposed in [1] (see Fig. 2), where in addition to the power-law degree distribution, the node clustering coefficients also exhibits a "scaling law" as a function of the node degree. In Figs. 4(a) and 4(b), we plot the CCDF of the degree distribution $X := k_i$ and the *clustering coefficient inverse* (CCI), $Y = 1/C_i$, for all nodes i's in the log-log scale. The linearity of both plots indicates that both (marginal) distributions are indeed regularly varying (i.e., are heavy-tailed). Looking at the *joint degree-CCI* distribution, we define and compute $Z(r) := \{X^{\alpha} r^{\beta}, Y^{\beta}\}$ (where the tail indices α and β are estimated from Figs. 4(a) and 4(b)) for a range of r values (see the right bar in Fig. 4(c)). Using the method presented at the end of Section 3 (the plots in this section are best viewed *in color*), we plot the CCDF of Z's in the log-log scale for this range of r values. The linear behavior of CCDFs for a wide range of different ratios r indicates that the joint degree-CCI distribution is *bivariate heavy tailed*, capturing the "hierarchical" relation between the high-degree center nodes and the "modules" in the recursive construction of the network model in [1].



(a) Ravasz model. (b) Dorogovtsev *et al.*model

Fig. 6: MRV Analysis for Joint Degree-Degree Distributions

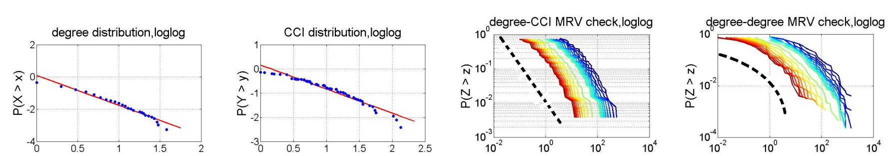
Applying the same MRV analysis to the "hierarchical" network model proposed in [5] (see Fig.3), we show that the joint degree-CCI distribution of this model is also bivariated heavy-tailed: Figs.5(a) and 5(b) show the marginals are regularly varying, while Fig.5(c) verifies the existence of MRV. Comparing these two deterministic network models, besides their "hierarchy" (in the sense of [1]), the network models have very different structural properties. For instance, Dorogovtsev's model [5] grows hierarchically by "glueing" the smaller models at the high-degree nodes ("hubs") to form a larger cluster and repeating this process recursively. In contrast, Ravasz's model [1] grows hierarchically by recursively attaching the *peripheral nodes* of the non-central clusters to the *hub* of the central cluster. The distinction between these two models can be revealed when we examine and perform the MRV analysis on the *joint degree-degree* distributions[2] as shown in Figs. 6(a) and 6(b). The clear non-linearity of the "contours" in Fig. 6(a) indicates the lack of MRV for degree-degree pairs in Ravasz's model [1], whereas the joint degree-degree distribution in Dorogovtsev's model [5] is bivariate heavy-tailed.



(a) Mariginal deg distr. (b) Mariginal CCI distr. (c) Joint deg-CCI distr. (d) Joint degree-degree distr.

Fig. 7: MRV Analysis for the Internet AS dataset.

That MRV analysis captures the *common* as well as *distinctive* structural characteristics of these two models illustrates the ability of the proposed MRV framework as a tool to help us better analyze and understand the *(hierarchical) structure* of complex networks: both networks are "hierarchical" in the sense of [1], but they differ in their "nature" of *hierarchy* (or near "self-similarity"). In addition, we note that both networks have similar marginal features, such as power-law marginal degree and CCI distributions as well as the "small-world" property. This signifies that marginals alone cannot capture the *dependence structure* among various constituting network "modules" and thus are not as informative as when examining in a *multivariate* context.



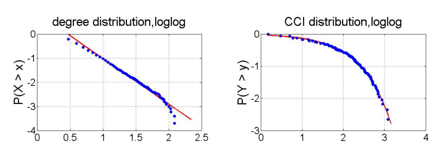
(a) Mariginal deg distr. (b) Mariginal CCI distr. (c) Joint deg-CCI distr. (d) Joint degree-degree distr.

Fig. 8: MRV Analysis for the PIN dataset.

[2] Joint degree-degree pairs are defined for the edges of a network, where each degree belongs to one endpoint of an edge.

We have also performed MRV analysis to other network models proposed in the literature. For example, we find that the deterministic network model *with zero clustering coefficients* proposed in [6] contains a bivariate heavy-tailed *degree-degree* distribution. This captures part of the "modular" structure built in the recursive construction process similar to that of [5], but using squares instead of triangles. (The authors claim that their network model is also "hierarchical" as a contradition to the definition/claim made in [1]). In fact, we believe that instead of measuring "modularity" in terms of clustering coefficients, if one uses a generalized modularity metric (e.g., counting the number of squares a node's neighbors are in), this network model will likely exhibits a mutivariate heavy tail in terms of degree and this generalized modularity metric. We also show that the "self-similar" or "fractal" networks as defined and identified by Song *et al.*in [2][3] (e.g., the PIN networks discussed below) contain MRVs in one form or another (e.g., joint degree-CCI distribution). Moreover, scale-free networks with self-similarity patterns suggested by high s-metric value [11], tend to show MRV in joint degree-degree distribution. Due to space limitation, we do not delve into details.

Last but not the least, we have also performed the MRV analysis on number of real network datasets. Due to space limitation, we provide only two examples: Internet at Autonomous System (AS) level and Protein-Protein Interaction (PIN) networks. These networks have been characterized as "hierarchical" [1, 13] or "self-similar" [2, 3, 10]. The log-log plots of the marginal degree and CCI and joint degree-CCI as well as the joint degree-degree distribution for the AS network are shown in Figs.7 (a)-(d) (for the same range of r values as before, which are not shown in Figs.7(c) & 7(d)). Those for the PIN network are shown in Figs.8 (a)-(d). We see that all mariginals are (approximately) regularly varying. The "contours" in the log-log plots for the degree-CCI distributions for both networks are fairly linear (with the AS network showing stronger linearity in the "tails"), suggestive of a bivariate heavy tail. On the other hand, the degree-degree of the AS network contains a clear bivariate heavy tail, while that of the PIN network lacks a clear bivariate heavy tail. At the end, we provide the MRV analysis for a random network generated by PA method to emphasize that MRV properties appear only in "hierarchical" or "self-similar" networks. Fig. (9(a),9(b)) shows the marginals for degree and CCI of this network indicating the lack of even the *necessary* conditions for MRV in degree-CCI (CCI is not heavy-tailed). This network does not show MRV properties for the other metrics either.



(a) Mariginal deg distr.

(b) Mariginal CCI distr.

Fig. 9: marginals for the PA random network

5 CONCLUSION

We have advocated a *multivariate analysis* perspective based on the notion of MRVs as a unifying framework to study complex structures in networks. Applying the theory of MRVs to complex network analysis, we have demonstrated the existence of "multivariate heavy tails" in synthetic and real networks. Our analysis also poses a number of new research questions such as how to best characterize hierarchical or modular structures in complex networks. Answering these questions are part of ongoing research. To the best of our knowledge, we believe ours is the first to apply such a framework in the study of complex networks, and demonstrate the existence of "multivariate heavy tails" in synthetic and real networks.

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